CHAPTER 4

Trigonometry and the Unit Circle

Have you ever wondered about the repeating patterns that occur around us? Repeating patterns occur in sound, light, tides, time, and molecular motion. To analyse these repeating, cyclical patterns, you need to move from using ratios in triangles to using circular functions to approach trigonometry.

In this chapter, you will learn how to model and solve trigonometric problems using the unit circle and circular functions of radian measures.

Did You Know?

The flower in the photograph is called the Trigonometry daffodil. Why do you think this name was chosen?

Key Terms

radian	cosecant
coterminal angles	secant
general form	cotangent
unit circle	trigonometric equation

Career Link

Engineers, police investigators, and legal experts all play key roles following a serious collision. Investigating and analysing a motor vehicle collision can provide valuable evidence for police and insurance reports. You can be trained in this fascinating and important field at police schools, engineering departments, and technical institutes.

Web Link

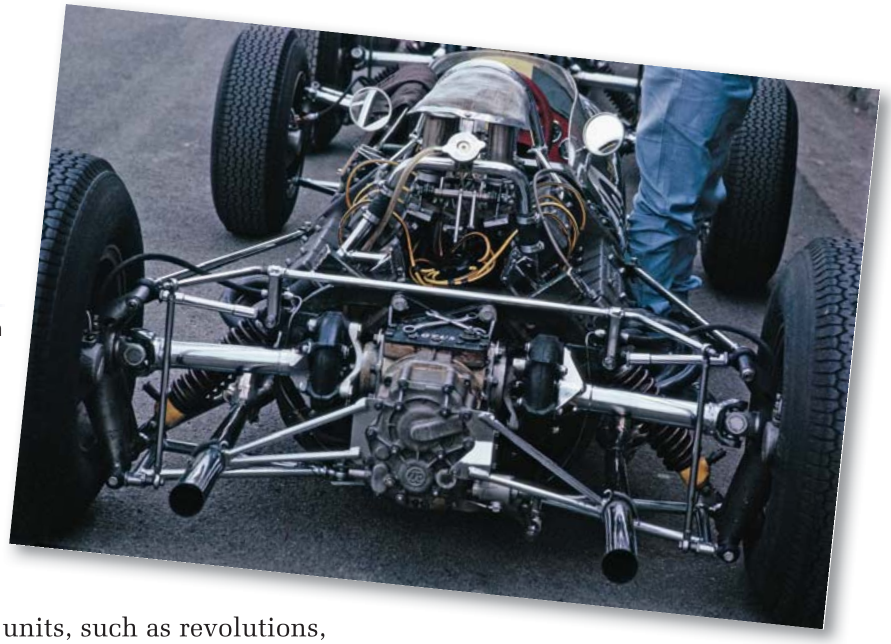
To learn more about accident reconstruction and training to become a forensic analysis investigator, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

4.1

Angles and Angle Measure

Focus on…

- sketching angles in standard position measured in degrees and radians
- converting angles in degree measure to radian measure and vice versa
- determining the measures of angles that are coterminal with a given angle
- solving problems involving arc lengths, central angles, and the radius in a circle

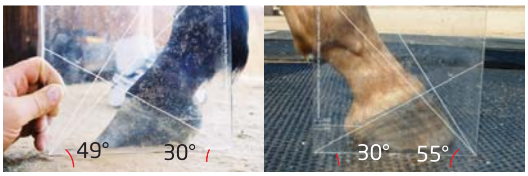


Angles can be measured using different units, such as revolutions, degrees, radians, and gradians. Which of these units are you familiar with? Check how many of these units are on your calculator.

Angles are everywhere and can be found in unexpected places. How many different angles can you see in the structure of the racing car?

Did You Know?

Sound (undamaged) hooves of all horses share certain angle aspects determined by anatomy and the laws of physics. A front hoof normally exhibits a 30° hairline and a 49° toe angle, while a hind hoof has a 30° hairline and a 55° toe angle.



Investigate Angle Measure

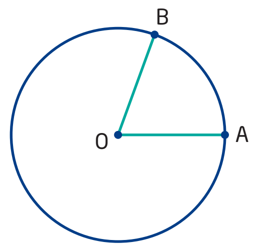
Materials

- masking tape
- sidewalk chalk
- string
- measuring tape

Work in small groups.

1. Mark the centre of a circle on the floor with sidewalk chalk. Then, using a piece of string greater than 1 m long for the radius, outline the circle with chalk or pieces of masking tape.
2. Label the centre of the circle O. Choose any point A on the circumference of the circle. OA is the radius of the circle. Have one member of your group walk heel-to-toe along the radius, counting foot lengths. Then, have the same person count the same number of foot lengths moving counterclockwise from A along the circumference. Label the endpoint B. Use tape to make the radii AO and BO. Have another member of the group confirm that the radius AO is the same length as arc AB.

3. Determine, by walking round the circle from B, approximately how many times the length of the radius fits onto the circumference.



Reflect and Respond

4. Use your knowledge of circumference to show that your answer in step 3 is reasonable.
5. Is $\angle AOB$ in step 3 greater than, equal to, or less than 60°? Discuss this with your group.
6. Determine the degree measure of $\angle AOB$, to the nearest tenth of a degree.
7. Compare your results with those of other groups. Does the central angle AOB maintain its size if you use a larger circle? a smaller circle?

Link the Ideas

In the investigation, you encountered several key points associated with angle measure.

By convention, angles measured in a counterclockwise direction are said to be positive. Those measured in a clockwise direction are negative.

The angle AOB that you created measures 1 **radian**.

One full rotation is 360° or 2π radians.

One half rotation is 180° or π radians.

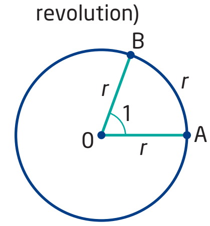
One quarter rotation is 90° or $\frac{\pi}{2}$ radians.

One eighth rotation is 45° or $\frac{\pi}{4}$ radians.

Many mathematicians omit units for radian measures. For example, $\frac{2\pi}{3}$ radians may be written as $\frac{2\pi}{3}$. Angle measures without units are considered to be in radians.

radian

- one radian is the measure of the central angle subtended in a circle by an arc equal in length to the radius of the circle
- $2\pi = 360°$
  $= 1$ full rotation (or revolution)



Example 1

Convert Between Degree and Radian Measure

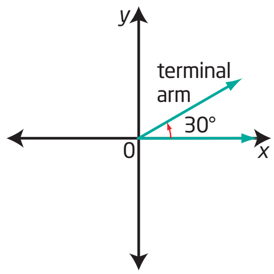
Draw each angle in standard position. Change each degree measure to radian measure and each radian measure to degree measure. Give answers as both exact and approximate measures (if necessary) to the nearest hundredth of a unit.

a) $30°$ **b)** $-120°$

c) $\frac{5\pi}{4}$ **d)** 2.57

Solution

a)



An angle in standard position has its centre at the origin and its initial arm along the positive *x*-axis.

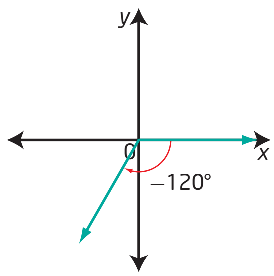
In which direction are positive angles measured?

Unitary Method

$$\begin{aligned} 360° &= 2\pi \\ 1° &= \frac{2\pi}{360} \\ &= \frac{\pi}{180} \\ 30° &= 30\left(\frac{\pi}{180}\right) \\ &= \frac{\pi}{6} \\ &\approx 0.52 \end{aligned}$$

$\frac{\pi}{6}$ is an exact value.

b)



Why is the angle drawn using a clockwise rotation?

Proportion Method

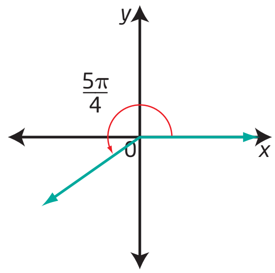
$$\begin{aligned} 180° &= \pi \\ \frac{-120°}{180°} &= \frac{x}{\pi} \\ x &= \frac{-120\pi}{180} \\ &= -\frac{2\pi}{3} \\ &\approx -2.09 \end{aligned}$$

So, $-120°$ is equivalent to $-\frac{2\pi}{3}$ or approximately -2.09.

c) π is $\frac{1}{2}$ rotation.

$\frac{\pi}{4}$ is $\frac{1}{8}$ rotation.

So $\frac{5\pi}{4}$ terminates in the third quadrant.



Unit Analysis

$$\frac{5\pi}{4} = \left(\frac{5\pi}{4}\right)\left(\frac{180°}{\pi}\right)$$
$$= \frac{5(180°)}{4}$$
$$= 225°$$

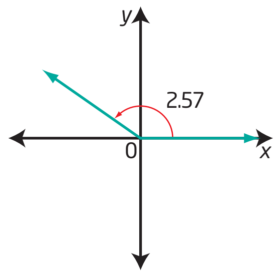
$\frac{5\pi}{4}$ is equivalent to 225°.

Why does $\left(\frac{180°}{\pi}\right)$ have value 1?

d) π (approximately 3.14) is $\frac{1}{2}$ rotation.

$\frac{\pi}{2}$ (approximately 1.57) is $\frac{1}{4}$ rotation.

2.57 is between 1.57 and 3.14, so it terminates in the second quadrant.



Unitary Method

$$\pi = 180°$$
$$1 = \frac{180°}{\pi}$$
$$2.57$$
$$= 2.57\left(\frac{180°}{\pi}\right)$$
$$= \frac{462.6°}{\pi}$$
$$\approx 147.25°$$

Proportion Method

$$\frac{x}{2.57} = \frac{180°}{\pi}$$
$$x = 2.57\left(\frac{180°}{\pi}\right)$$
$$x = \frac{462.6°}{\pi}$$
$$x \approx 147.25°$$

Unit Analysis

$$2.57$$
$$= 2.57\left(\frac{180°}{\pi}\right)$$
$$= \frac{462.6°}{\pi}$$
$$\approx 147.25°$$

2.57 is equivalent to $\frac{462.6°}{\pi}$ or approximately 147.25°.

Your Turn

Draw each angle in standard position. Change each degree measure to radians and each radian measure to degrees. Give answers as both exact and approximate measures (if necessary) to the nearest hundredth of a unit.

a) $-270°$

b) 150°

c) $\frac{7\pi}{6}$

d) -1.2

Did You Know?

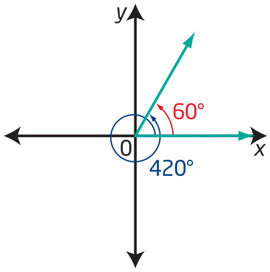
Most scientific and graphing calculators can calculate using angle measures in both degrees and radians. Find out how to change the mode on your calculator.

Coterminal Angles

When you sketch an angle of 60° and an angle of 420° in standard position, the terminal arms coincide. These are **coterminal angles**.

coterminal angles

- angles in standard position with the same terminal arms
- may be measured in degrees or radians
- $\frac{\pi}{4}$ and $\frac{9\pi}{4}$ are coterminal angles, as are 40° and −320°



Example 2

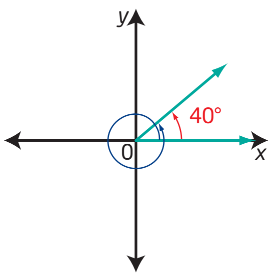
Identify Coterminal Angles

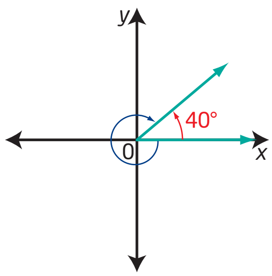
Determine one positive and one negative angle measure that is coterminal with each angle. In which quadrant does the terminal arm lie?

a) 40° **b)** −430° **c)** $\frac{8\pi}{3}$

Solution

a) The terminal arm is in quadrant I.
To locate coterminal angles, begin on the terminal arm of the given angle and rotate in a positive or negative direction until the new terminal arm coincides with that of the original angle.



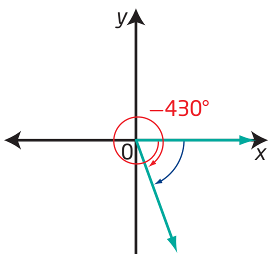


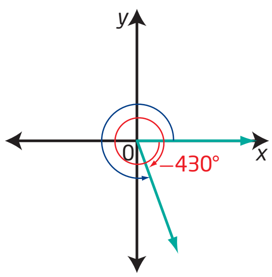
$40° + 360° = 400°$ $40° + (-360°) = -320°$

Two angles coterminal with 40° are 400° and −320°.

What other answers are possible?

b) The terminal arm of −430° is in quadrant IV.

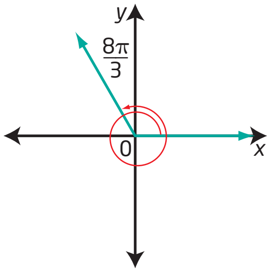




$-430° + 360° = -70°$ $-430° + 720° = 290°$

Two angles coterminal with −430° are 290° and −70°.

The reference angle is 70°.

c) 

$\frac{8\pi}{3} = \frac{6\pi}{3} + \frac{2\pi}{3}$

So, the angle is one full rotation (2π) plus $\frac{2\pi}{3}$.

The terminal arm is in quadrant II.

There are 2π or $\frac{6\pi}{3}$ in one full rotation.

Counterclockwise one full rotation: $\frac{8\pi}{3} + \frac{6\pi}{3} = \frac{14\pi}{3}$

Clockwise one full rotation: $\frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$

Clockwise two full rotations: $\frac{8\pi}{3} - \frac{12\pi}{3} = -\frac{4\pi}{3}$

Two angles coterminal with $\frac{8\pi}{3}$ are $\frac{2\pi}{3}$ and $-\frac{4\pi}{3}$.

Your Turn

For each angle in standard position, determine one positive and one negative angle measure that is coterminal with it.

a) $270°$ **b)** $-\frac{5\pi}{4}$ **c)** $740°$

Coterminal Angles in General Form

By adding or subtracting multiples of one full rotation, you can write an infinite number of angles that are coterminal with any given angle.

For example, some angles that are coterminal with 40° are

$40° + (360°)(1) = 400°$ $\quad$ $40° - (360°)(1) = -320°$

$40° + (360°)(2) = 760°$ $\quad$ $40° - (360°)(2) = -680°$

In general, the angles coterminal with 40° are $40° \pm (360°)n$, where n is any natural number.

Some angles coterminal with $\frac{2\pi}{3}$ are

$$\begin{aligned}\frac{2\pi}{3} + 2\pi(1) &= \frac{2\pi}{3} + \frac{6\pi}{3} \\ &= \frac{8\pi}{3}\end{aligned} \qquad \begin{aligned}\frac{2\pi}{3} - 2\pi(1) &= \frac{2\pi}{3} - \frac{6\pi}{3} \\ &= -\frac{4\pi}{3}\end{aligned}$$

$$\begin{aligned}\frac{2\pi}{3} + 2\pi(2) &= \frac{2\pi}{3} + \frac{12\pi}{3} \\ &= \frac{14\pi}{3}\end{aligned} \qquad \begin{aligned}\frac{2\pi}{3} - 2\pi(2) &= \frac{2\pi}{3} - \frac{12\pi}{3} \\ &= -\frac{10\pi}{3}\end{aligned}$$

In general, the angles coterminal with $\frac{2\pi}{3}$ are $\frac{2\pi}{3} \pm 2\pi n$, where n is any natural number.

Any given angle has an infinite number of angles coterminal with it, since each time you make one full rotation from the terminal arm, you arrive back at the same terminal arm. Angles coterminal with any angle θ can be described using the expression

$\theta \pm (360°)n$ or $\theta \pm 2\pi n$,

where n is a natural number. This way of expressing an answer is called the **general form**.

general form

- an expression containing parameters that can be given specific values to generate any answer that satisfies the given information or situation
- represents all possible cases

Example 3

Express Coterminal Angles in General Form

a) Express the angles coterminal with 110° in general form. Identify the angles coterminal with 110° that satisfy the domain $-720° \leq \theta < 720°$.

b) Express the angles coterminal with $\frac{8\pi}{3}$ in general form. Identify the angles coterminal with $\frac{8\pi}{3}$ in the domain $-4\pi \leq \theta < 4\pi$.

Solution

a) Angles coterminal with 110° occur at $110° \pm (360°)n$, $n \in \mathrm{N}$.

Substitute values for n to determine these angles.

n	1	2	3
$110° - (360°)n$	−250°	−610°	−970°
$110° + (360°)n$	470°	830°	1190°

From the table, the values that satisfy the domain $-720° \leq \theta < 720°$ are −610°, −250°, and 470°. These angles are coterminal.

b) $\frac{8\pi}{3} \pm 2\pi n$, $n \in \mathrm{N}$, represents all angles coterminal with $\frac{8\pi}{3}$. Substitute values for n to determine these angles.

n	1	2	3	4
$\frac{8\pi}{3} - 2\pi n$	$\frac{2\pi}{3}$	$-\frac{4\pi}{3}$	$-\frac{10\pi}{3}$	$-\frac{16\pi}{3}$
$\frac{8\pi}{3} + 2\pi n$	$\frac{14\pi}{3}$	$\frac{20\pi}{3}$	$\frac{26\pi}{3}$	$\frac{32\pi}{3}$

The angles in the domain $-4\pi \leq \theta < 4\pi$ that are coterminal are $-\frac{10\pi}{3}$, $-\frac{4\pi}{3}$, and $\frac{2\pi}{3}$.

Why is $-\frac{16\pi}{3}$ not an acceptable answer?

Your Turn

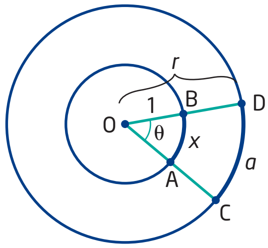
Write an expression for all possible angles coterminal with each given angle. Identify the angles that are coterminal that satisfy $-360° \leq \theta < 360°$ or $-2\pi \leq \theta < 2\pi$.

a) −500° **b)** 650° **c)** $\frac{9\pi}{4}$

Arc Length of a Circle

All arcs that subtend a right angle $\left(\frac{\pi}{2}\right)$ have the same central angle, but they have different arc lengths depending on the radius of the circle. The arc length is proportional to the radius. This is true for any central angle and related arc length.

Consider two concentric circles with centre O. The radius of the smaller circle is 1, and the radius of the larger circle is r. A central angle of θ radians is subtended by arc AB on the smaller circle and arc CD on the larger one. You can write the following proportion, where x represents the arc length of the smaller circle and a is the arc length of the larger circle.



$$\frac{a}{x} = \frac{r}{1}$$
$$a = xr \qquad ①$$

Consider the circle with radius 1 and the sector with central angle θ. The ratio of the arc length to the circumference is equal to the ratio of the central angle to one full rotation.

$$\frac{x}{2\pi r} = \frac{\theta}{2\pi}$$
Why is $r = 1$?
$$x = \left(\frac{\theta}{2\pi}\right)2\pi(1)$$
$$x = \theta$$

Substitute $x = \theta$ in ①.

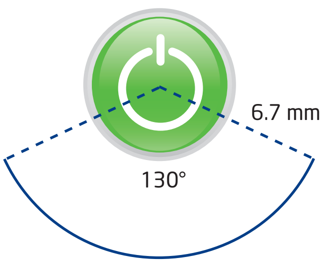
$a = \theta r$

This formula, $a = \theta r$, works for any circle, provided that θ is measured in radians and both a and r are measured in the same units.

Example 4

Determine Arc Length in a Circle

Rosemarie is taking a course in industrial engineering. For an assignment, she is designing the interface of a DVD player. In her plan, she includes a decorative arc below the on/off button. The arc has central angle 130° in a circle with radius 6.7 mm. Determine the length of the arc, to the nearest tenth of a millimetre.



Solution

Method 1: Convert to Radians and Use the Formula $a = \theta r$
Convert the measure of the central angle to radians before using the formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the length of the radius.

$$180° = \pi$$
$$1° = \frac{\pi}{180}$$
$$130° = 130\left(\frac{\pi}{180}\right)$$
$$= \frac{13\pi}{18}$$

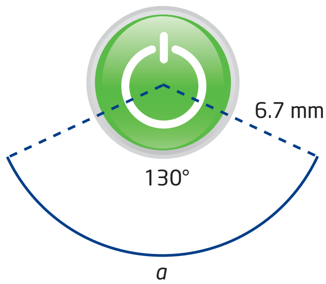
$$a = \theta r$$
$$= \left(\frac{13\pi}{18}\right)(6.7)$$
$$= \frac{87.1\pi}{18}$$
$$= 15.201\ldots$$

Why is it important to use exact values throughout the calculation and only convert to decimal fractions at the end?

The arc length is 15.2 mm, to the nearest tenth of a millimetre.

Method 2: Use a Proportion
Let a represent the arc length.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\text{central angle}}{\text{full rotation}}$$
$$\frac{a}{2\pi(6.7)} = \frac{130°}{360°}$$
$$a = \frac{2\pi(6.7)130°}{360°}$$
$$= 15.201\ldots$$



The arc length is 15.2 mm, to the nearest tenth of a millimetre.

Your Turn

If a represents the length of an arc of a circle with radius r, subtended by a central angle of θ, determine the missing quantity. Give your answers to the nearest tenth of a unit.

a) $r = 8.7$ cm, $\theta = 75°$, $a = \blacksquare$ cm
b) $r = \blacksquare$ mm, $\theta = 1.8$, $a = 4.7$ mm
c) $r = 5$ m, $a = 13$ m, $\theta = \blacksquare$

Key Ideas

- Angles can be measured using different units, including degrees and radians.
- An angle measured in one unit can be converted to the other unit using the relationships 1 full rotation $= 360° = 2\pi$.
- An angle in standard position has its vertex at the origin and its initial arm along the positive x-axis.
- Angles that are coterminal have the same initial arm and the same terminal arm.
- An angle θ has an infinite number of angles that are coterminal expressed by $\theta \pm (360°)n$, $n \in N$, in degrees, or $\theta \pm 2\pi n$, $n \in N$, in radians.
- The formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the length of the radius, can be used to determine any of the variables given the other two, as long as a and r are in the same units.

Check Your Understanding

Practise

1. For each angle, indicate whether the direction of rotation is clockwise or counterclockwise.

a) -4π **b)** $750°$

c) $-38.7°$ **d)** 1

2. Convert each degree measure to radians. Write your answers as exact values. Sketch the angle and label it in degrees and in radians.

a) $30°$ **b)** $45°$

c) $-330°$ **d)** $520°$

e) $90°$ **f)** $21°$

3. Convert each degree measure to radians. Express your answers as exact values and as approximate measures, to the nearest hundredth of a radian.

a) $60°$ **b)** $150°$

c) $-270°$ **d)** $72°$

e) $-14.8°$ **f)** $540°$

4. Convert each radian measure to degrees. Express your answers as exact values and as approximate measures, to the nearest tenth of a degree, if necessary.

a) $\frac{\pi}{6}$ **b)** $\frac{2\pi}{3}$

c) $-\frac{3\pi}{8}$ **d)** $-\frac{5\pi}{2}$

e) 1 **f)** 2.75

5. Convert each radian measure to degrees. Express your answers as exact values and as approximate measures, to the nearest thousandth.

a) $\frac{2\pi}{7}$ **b)** $\frac{7\pi}{13}$

c) $\frac{2}{3}$ **d)** 3.66

e) -6.14 **f)** -20

6. Sketch each angle in standard position. In which quadrant does each angle terminate?

a) 1 **b)** $-225°$

c) $\frac{17\pi}{6}$ **d)** $650°$

e) $-\frac{2\pi}{3}$ **f)** $-42°$

7. Determine one positive and one negative angle coterminal with each angle.

a) $72°$
b) $\frac{3\pi}{4}$
c) $-120°$
d) $\frac{11\pi}{2}$
e) $-205°$
f) 7.8

8. Determine whether the angles in each pair are coterminal. For one pair of angles, explain how you know.

a) $\frac{5\pi}{6}, \frac{17\pi}{6}$
b) $\frac{5\pi}{2}, -\frac{9\pi}{2}$
c) $410°, -410°$
d) $227°, -493°$

9. Write an expression for all of the angles coterminal with each angle. Indicate what your variable represents.

a) $135°$
b) $-\frac{\pi}{2}$
c) $-200°$
d) 10

10. Draw and label an angle in standard position with negative measure. Then, determine an angle with positive measure that is coterminal with your original angle. Show how to use a general expression for coterminal angles to find the second angle.

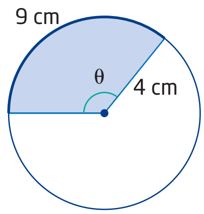
11. For each angle, determine all angles that are coterminal in the given domain.

a) $65°, 0° \leq \theta < 720°$
b) $-40°, -180° \leq \theta < 360°$
c) $-40°, -720° \leq \theta < 720°$
d) $\frac{3\pi}{4}, -2\pi \leq \theta < 2\pi$
e) $-\frac{11\pi}{6}, -4\pi \leq \theta < 4\pi$
f) $\frac{7\pi}{3}, -2\pi \leq \theta < 4\pi$
g) $2.4, -2\pi \leq \theta < 2\pi$
h) $-7.2, -4\pi \leq \theta < 2\pi$

12. Determine the arc length subtended by each central angle. Give answers to the nearest hundredth of a unit.

a) radius 9.5 cm, central angle 1.4
b) radius 1.37 m, central angle 3.5
c) radius 7 cm, central angle 130°
d) radius 6.25 in., central angle 282°

13. Use the information in each diagram to determine the value of the variable. Give your answers to the nearest hundredth of a unit.

a)


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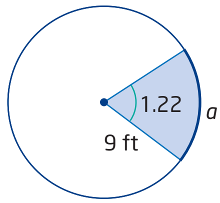

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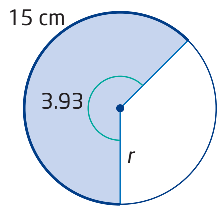

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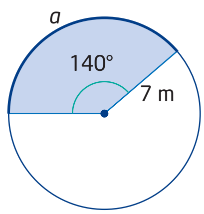


Apply

14. A rotating water sprinkler makes one revolution every 15 s. The water reaches a distance of 5 m from the sprinkler.

a) What is the arc length of the sector watered when the sprinkler rotates through $\frac{5\pi}{3}$? Give your answer as both an exact value and an approximate measure, to the nearest hundredth.

b) Show how you could find the area of the sector watered in part a).

c) What angle does the sprinkler rotate through in 2 min? Express your answer in radians and degrees.

15. Angular velocity describes the rate of change in a central angle over time. For example, the change could be expressed in revolutions per minute (rpm), radians per second, degrees per hour, and so on. All that is required is an angle measurement expressed over a unit of time.

a) Earth makes one revolution every 24 h. Express the angular velocity of Earth in three other ways.

b) An electric motor rotates at 1000 rpm. What is this angular velocity expressed in radians per second?

c) A bicycle wheel completes 10 revolutions every 4 s. Express this angular velocity in degrees per minute.

16. Skytrek Adventure Park in Revelstoke, British Columbia, has a sky swing. Can you imagine a 170-ft flight that takes riders through a scary pendulum swing? At one point you are soaring less than 10 ft from the ground at speeds exceeding 60 mph.

a) The length of the cable is 72 ft and you travel on an arc of length 170 ft on one particular swing. What is the measure of the central angle? Give your answer in radians, to the nearest hundredth.

b) What is the measure of the central angle from part a), to the nearest tenth of a degree?

17. Copy and complete the table by converting each angle measure to its equivalent in the other systems. Round your answers to the nearest tenth where necessary.

	Revolutions	Degrees	Radians
a)	1 rev		
b)		270°	
c)			$\frac{5\pi}{6}$
d)			−1.7
e)		−40°	
f)	0.7 rev		
g)	−3.25 rev		
h)		460°	
i)			$-\frac{3\pi}{8}$

18. Joran and Jasmine are discussing expressions for the general form of coterminal angles of 78°. Joran claims the answer must be expressed as $78° + (360°)n$, $n \in I$. Jasmine indicates that although Joran's expression is correct, another answer is possible. She prefers $78° \pm k(360°)$, $k \in N$, where N represents positive integers. Who is correct? Why?

19. The gradian (grad) is another unit of angle measure. It is defined as $\frac{1}{400}$ of a revolution, so one full rotation contains 400 grads.

a) Determine the number of gradians in 50°.

b) Describe a process for converting from degree measure to gradians and vice versa.

c) Identify a possible reason that the gradian was created.

Did You Know?

Gradians originated in France in the 1800s. They are still used in some engineering work.

20. Yellowknife, Northwest Territories, and Crowsnest Pass, Alberta, lie along the 114° W line of longitude. The latitude of Yellowknife is 62.45° N and the latitude of Crowsnest Pass is 49.63° N. Consider Earth to be a sphere with radius 6400 km.

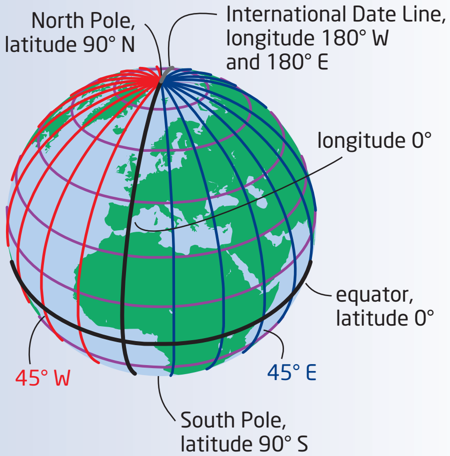
a) Sketch the information given above using a circle. Label the centre of Earth, its radius to the equator, and the locations of Yellowknife and Crowsnest Pass.

b) Determine the distance between Yellowknife and Crowsnest Pass. Give your answer to the nearest hundredth of a kilometre.

c) Choose a town or city either where you live or nearby. Determine the latitude and longitude of this location. Find another town or city with the same longitude. What is the distance between the two places?

Did You Know?

Lines of latitude and longitude locate places on Earth. Lines of latitude are parallel to the equator and are labelled from 0° at the equator to 90° at the North Pole. Lines of longitude converge at the poles and are widest apart at the equator. 0° passes through Greenwich, England, and the lines are numbered up to 180° E and 180° W, meeting at the International Date Line.



21. Sam Whittingham from Quadra Island, British Columbia, holds five 2009 world human-powered speed records on his recumbent bicycle. In the 200-m flying start, he achieved a speed of 133.284 km/h.

a) Express the speed in metres per minute.

b) The diameter of his bicycle wheel is 60 cm. Through how many radians per minute must the wheels turn to achieve his world record in the 200-m flying start?

22. A water wheel with diameter 3 m is used to measure the approximate speed of the water in a river. If the angular velocity of the wheel is 15 rpm, what is the speed of the river, in kilometres per hour?

23. Earth is approximately 93 000 000 mi from the sun. It revolves around the sun, in an almost circular orbit, in about 365 days. Calculate the linear speed, in miles per hour, of Earth in its orbit. Give your answer to the nearest hundredth.

Extend

24. Refer to the Did You Know? below.

a) With a partner, show how to convert 69.375° to 69° 22′ 30″.

b) Change the following angles into degrees-minutes-seconds.

i) 40.875° **ii)** 100.126°
iii) 14.565° **iv)** 80.385°

Did You Know?

You have expressed degree measures as decimal numbers, for example, 69.375°. Another way subdivides 1° into 60 parts called minutes. Each minute can be subdivided into 60 parts called seconds. Then, an angle such as 69.375° can be written as 69° 22 min 30 s or 69° 22′ 30″.

25. a) Reverse the process of question 24 and show how to convert 69° 22′ 30″ to 69.375°. Hint: Convert 30″ into a decimal fraction part of a minute. Combine this part of a minute with the 22′ and then convert the minutes to part of a degree.

b) Change each angle measure into degrees, rounded to the nearest thousandth.

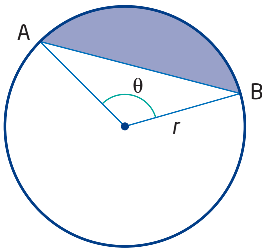
i) 45° 30′ 30″

ii) 72° 15′ 45″

iii) 105° 40′ 15″

iv) 28° 10′

26. A segment of a circle is the region between a chord and the arc subtended by that chord. Consider chord AB subtended by central angle θ in a circle with radius *r*. Derive a formula using *r* and θ for the area of the segment subtended by θ.



27. The hour hand of an analog clock moves in proportion to the movement of the minute hand. This means that at 4:05, the hour hand will have moved beyond the 4 by $\frac{5}{60}$ of the distance it would move in an hour.

a) What is the measure of the obtuse angle between the hands of a clock at 4:00? Give your answer in degrees.

b) What is the measure, in degrees, of the acute angle between the hands of a clock at 4:10?

c) At certain times, the hands of a clock are at right angles to each other. What are two of these times?

d) At how many different times does the angle between the hands of a clock measure 90° between 4:00 and 5:00?

e) Does one of the times occur before, at, or shortly after 4:05? Explain.

Create Connections

C1 Draw a diagram and use it to help explain whether 6 radians is greater than, equal to, or less than 360°.

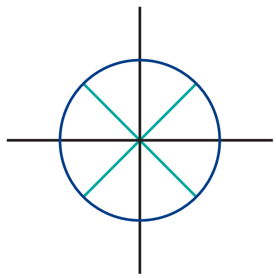
C2 In mathematics, angle measures are commonly defined in degrees or radians. Describe the difference between 1° and 1 radian. Use drawings to support your answer.

C3 The following angles are in standard position. What is the measure of the reference angle for each? Write an expression for all coterminal angles associated with each given angle.

a) 860°

b) −7 (give the reference angle to the nearest hundredth)

C4 a) Make a circle diagram similar to the one shown. On the outside of the circle, label all multiples of 45° in the domain $0° \leq \theta < 360°$. Show the radian equivalent as an exact value inside the circle.



b) Make another circle diagram. This time, mark and label all the multiples of 30° in the domain $0° \leq \theta < 360°$. Again, show the degree values outside the circle and the exact radian equivalents inside the circle.

C5 A line passes through the point (3, 0). Find the equation of the line if the angle formed between the line and the positive *x*-axis is

a) $\frac{\pi}{2}$

b) 45°

4.2

The Unit Circle

Focus on…

- developing and applying the equation of the unit circle
- generalizing the equation of a circle with centre (0, 0) and radius r
- using symmetry and patterns to locate the coordinates of points on the unit circle

A gauge is a measuring tool that is used in many different situations. There are two basic types of gauges—radial (circular) and linear. What gauges can you think of that are linear? What gauges are you familiar with that are circular? How are linear and circular gauges similar, and how do they differ?

Have you ever wondered why some phenomena, such as tides and hours of daylight, are so predictable? It is because they have repetitive or cyclical patterns. Why is sin 30° the same as sin 150°? Why is cos 60° = sin 150°? How do the coordinates of a point on a circle of radius 1 unit change every quarter-rotation?

Investigate Circular Number Lines

Materials

- paper
- scissors
- tape
- can or other cylinder
- straight edge
- compass

1. Select a can or other cylinder. Cut a strip of paper about 1.5 cm wide and the same length as the circumference of the cylinder.
2. Create a number line by drawing a line along the centre of the strip. Label the left end of the line 0 and the right end 2π. According to this labelling, how long is the number line?
3. Divide the number line into eight equal subdivisions. What value would you use to label the point midway between 0 and 2π? What value would you use to label halfway between 0 and the middle of the number line? Continue until all seven points that subdivide the number line are labelled. Write all values in terms of π. Express fractional values in lowest terms.
4. Tape the number line around the bottom of the can, so that the labels read in a counterclockwise direction.
5. Use the can to draw a circle on a sheet of paper. Locate the centre of the circle and label it O. Draw coordinate axes through O that extend beyond the circle. Place the can over the circle diagram so that the zero of the number line lies above where the circle intersects the positive x-axis.

6. Mark the coordinates of all points where the circle crosses the axes on your diagram. Label these points as $P(\theta) = (x, y)$, where $P(\theta)$ represents a point on the circle that has a central angle θ in standard position. For example, label the point where the circle crosses the positive y-axis as $P\left(\frac{\pi}{2}\right) = (0, 1)$.
7. Now, create a second number line. Label the ends as 0 and 2π. Divide this number line into 12 equal segments. Label divisions in terms of π. Express fractional values in lowest terms.

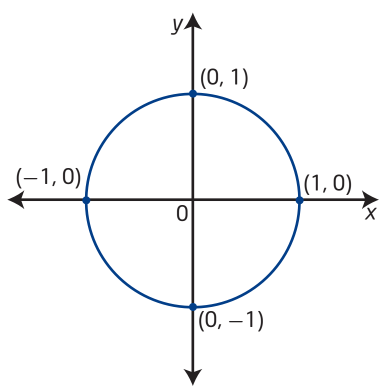
Reflect and Respond

8. Since each number line shows the circumference of the can and the circle to be 2π units, what assumption is being made about the length of the radius?
9. a) Two students indicate that the points in step 6 are simply multiples of $\frac{\pi}{2}$. Do you agree? Explain.
   b) In fact, they argue that the values on the original number line are all multiples of $\frac{\pi}{4}$. Is this true? Explain.
10. Show how to determine the coordinates for $P\left(\frac{\pi}{4}\right)$. Hint: Use your knowledge of the ratios of the side lengths of a 45°-45°-90° triangle. Mark the coordinates for all the points on the circle that are midway between the axes. What is the only difference in the coordinates for these four points? What negative values for θ would generate the same points on the circle midway between the axes?

Link the Ideas

Unit Circle

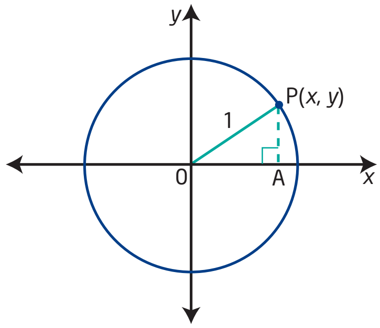
The circle you drew in the investigation is a **unit circle**.



unit circle
- a circle with radius 1 unit
- a circle of radius 1 unit with centre at the origin on the Cartesian plane is known as *the* unit circle

You can find the equation of the unit circle using the Pythagorean theorem.

Consider a point P on the unit circle.
Let P have coordinates (x, y).
Draw right triangle OPA as shown.



$OP = 1$ — The radius of the unit circle is 1.

$PA = |y|$ — The absolute value of the y-coordinate represents the distance from a point to the x-axis. Why is this true?

$OA = |x|$

$(OP)^2 = (OA)^2 + (PA)^2$ — Pythagorean theorem

$1^2 = |x|^2 + |y|^2$ — How would the equation for a circle with centre O(0, 0) differ if the radius were r rather than 1?

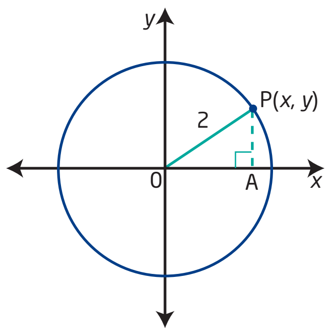
$1 = x^2 + y^2$

The equation of the unit circle is $x^2 + y^2 = 1$.

Example 1

Equation of a Circle Centred at the Origin

Determine the equation of the circle with centre at the origin and radius 2.



Solution

Choose a point, P, on the circle with coordinates (x, y).

The radius of the circle is 2, and a vertical line from the y-coordinate to the x-axis forms a right angle with the axis. This means you can use the Pythagorean theorem.

$|x|^2 + |y|^2 = 2^2$
$x^2 + y^2 = 4$

Since this is true for every point P on the circle, the equation of the circle is $x^2 + y^2 = 4$.

Your Turn

Determine the equation of a circle with centre at the origin and radius 6.

Example 2

Determine Coordinates for Points of the Unit Circle

Determine the coordinates for all points on the unit circle that satisfy the conditions given. Draw a diagram in each case.

a) the x-coordinate is $\frac{2}{3}$

b) the y-coordinate is $-\frac{1}{\sqrt{2}}$ and the point is in quadrant III

Solution

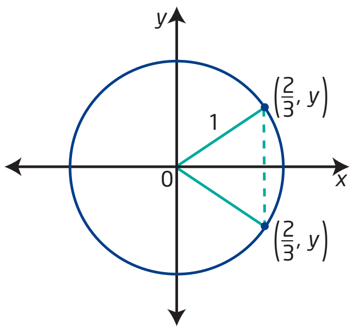
a) Coordinates on the unit circle satisfy the equation $x^2 + y^2 = 1$.

$$\left(\frac{2}{3}\right)^2 + y^2 = 1$$

Since x is positive, which quadrants could the points be in?

$$\frac{4}{9} + y^2 = 1$$
$$y^2 = \frac{5}{9}$$
$$y = \pm\frac{\sqrt{5}}{3}$$

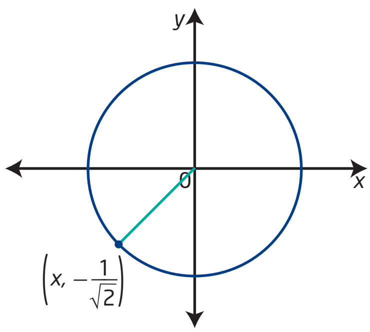
Why are there two answers?



Two points satisfy the given conditions: $\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$ in quadrant I and $\left(\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$ in quadrant IV.

b) $y = -\frac{1}{\sqrt{2}}$

y is negative in quadrants III and IV. But the point is in quadrant III, so x is also negative.



$$x^2 + y^2 = 1$$
$$x^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$$
$$x^2 + \frac{1}{2} = 1$$
$$x^2 = \frac{1}{2}$$
$$x = -\frac{1}{\sqrt{2}}$$

Why is there only one answer?

The point is $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, or $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Your Turn

Determine the missing coordinate(s) for all points on the unit circle satisfying the given conditions. Draw a diagram and tell which quadrant(s) the points lie in.

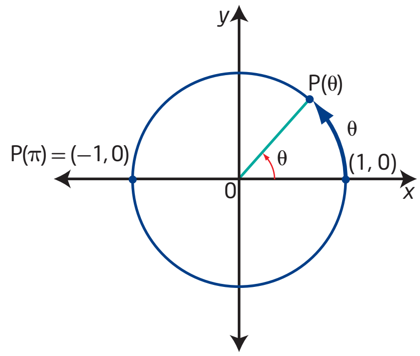
a) $\left(-\frac{5}{8}, y\right)$ **b)** $\left(x, \frac{5}{13}\right)$, where the point is in quadrant II

Relating Arc Length and Angle Measure in Radians

The formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the radius, applies to any circle, as long as a and r are measured in the same units. In the unit circle, the formula becomes $a = \theta(1)$ or $a = \theta$. This means that a central angle and its subtended arc on the unit circle have the same numerical value.

You can use the function $P(\theta) = (x, y)$ to link the arc length, θ, of a central angle in the unit circle to the coordinates, (x, y), of the point of intersection of the terminal arm and the unit circle.

If you join $P(\theta)$ to the origin, you create an angle θ in standard position. Now, θ radians is the central angle and the arc length is θ units.



Function P takes real-number values for the central angle or the arc length on the unit circle and matches them with specific points. For example, if $\theta = \pi$, the point is $(-1, 0)$. Thus, you can write $P(\pi) = (-1, 0)$.

Example 3

Multiples of $\frac{\pi}{3}$ on the Unit Circle

a) On a diagram of the unit circle, show the integral multiples of $\frac{\pi}{3}$ in the interval $0 \leq \theta \leq 2\pi$.

b) What are the coordinates for each point $P(\theta)$ in part a)?

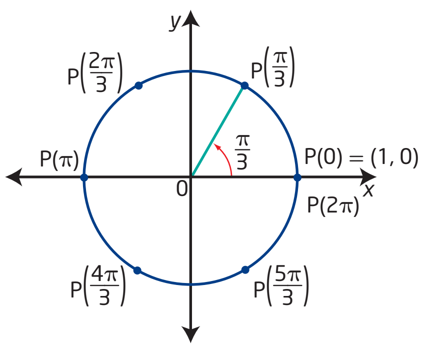
c) Identify any patterns you see in the coordinates of the points.

Solution

a) This is essentially a counting problem using $\frac{\pi}{3}$.

Multiples of $\frac{\pi}{3}$ in the interval $0 \leq \theta \leq 2\pi$ are

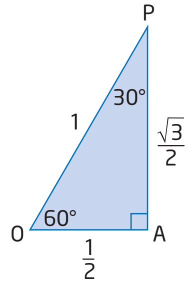
$0\left(\frac{\pi}{3}\right) = 0$, $1\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$, $2\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}$, $3\left(\frac{\pi}{3}\right) = \pi$, $4\left(\frac{\pi}{3}\right) = \frac{4\pi}{3}$, $5\left(\frac{\pi}{3}\right) = \frac{5\pi}{3}$, and $6\left(\frac{\pi}{3}\right) = 2\pi$.



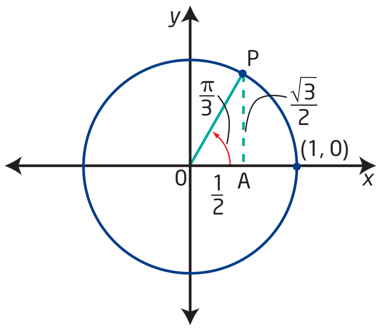
Why must you show only the multiples in one positive rotation in the unit circle?

b) Recall that a 30°-60°-90° triangle has sides in the ratio $1:\sqrt{3}:2$ or $\frac{1}{2}:\frac{\sqrt{3}}{2}:1$.

Place $\triangle$POA in the unit circle as shown.

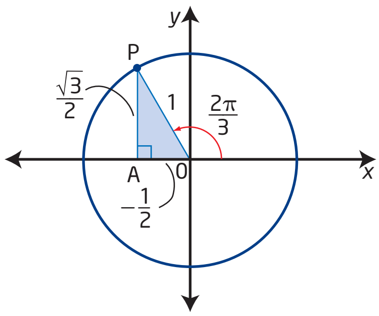


Why is the 30°-60°-90° triangle used?



Why are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ the coordinates of $P\left(\frac{\pi}{3}\right)$?

$\triangle$POA could be placed in the second quadrant with O at the origin and OA along the x-axis as shown. This gives $P\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



Why is the x-coordinate negative?

What transformation could be used to move $\triangle$POA from quadrant I to quadrant II?

Continue, placing $\triangle$POA in quadrants III and IV to find the coordinates of $P\left(\frac{4\pi}{3}\right)$ and $P\left(\frac{5\pi}{3}\right)$. Then, the coordinates of point P corresponding to angles that are multiples of $\frac{\pi}{3}$ are

$P(0) = P(2\pi) = (1, 0)$ $\quad$ $P(\pi) = (-1, 0)$ $\quad$ $P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$P\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\quad$ $P\left(\frac{4\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ $\quad$ $P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

c) Some patterns are:

- The points corresponding to angles that are multiples of $\frac{\pi}{3}$ that cannot be simplified, for example, $P\left(\frac{\pi}{3}\right)$, $P\left(\frac{2\pi}{3}\right)$, $P\left(\frac{4\pi}{3}\right)$, and $P\left(\frac{5\pi}{3}\right)$, have the same coordinates except for their signs.
- Any points where θ reduces to a multiple of π, for example, P(0), $P\left(\frac{3\pi}{3}\right) = P(\pi)$, and $P\left(\frac{6\pi}{3}\right) = P(2\pi)$, fall on an axis.

Your Turn

a) On a diagram of the unit circle, show all the integral multiples of $\frac{\pi}{6}$ in the interval $0 \le \theta < 2\pi$.

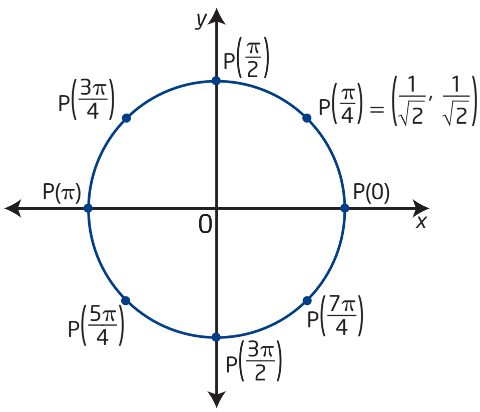
b) Label the coordinates for each point P(θ) on your diagram.

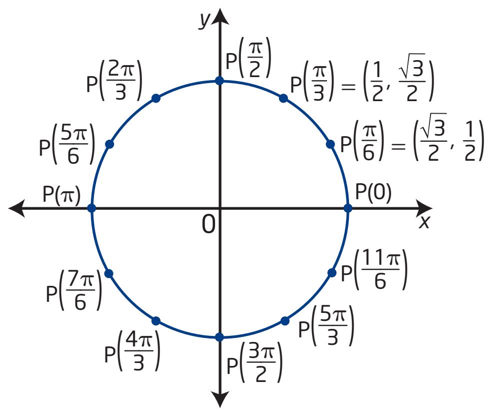
c) Describe any patterns you see in the coordinates of the points.

Key Ideas

- The equation for the unit circle is $x^2 + y^2 = 1$. It can be used to determine whether a point is on the unit circle or to determine the value of one coordinate given the other. The equation for a circle with centre at (0, 0) and radius r is $x^2 + y^2 = r^2$.
- On the unit circle, the measure in radians of the central angle and the arc subtended by that central angle are numerically equivalent.
- Some of the points on the unit circle correspond to exact values of the special angles learned previously.
- You can use patterns to determine coordinates of points. For example, the numerical value of the coordinates of points on the unit circle change to their opposite sign every $\frac{1}{2}$ rotation.

  If $P(\theta) = (a, b)$ is in quadrant I, then both a and b are positive. $P(\theta + \pi)$ is in quadrant III. Its coordinates are $(-a, -b)$, where $a > 0$ and $b > 0$.





Check Your Understanding

Practise

1. Determine the equation of a circle with centre at the origin and radius

a) 4 units

b) 3 units

c) 12 units

d) 2.6 units

2. Is each point on the unit circle? How do you know?

a) $\left(-\frac{3}{4}, \frac{1}{4}\right)$

b) $\left(\frac{\sqrt{5}}{8}, \frac{7}{8}\right)$

c) $\left(-\frac{5}{13}, \frac{12}{13}\right)$

d) $\left(\frac{4}{5}, -\frac{3}{5}\right)$

e) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

f) $\left(\frac{\sqrt{7}}{4}, \frac{3}{4}\right)$

3. Determine the missing coordinate(s) for all points on the unit circle satisfying the given conditions. Draw a diagram to support your answer.
   a) $\left(\frac{1}{4}, y\right)$ in quadrant I
   b) $\left(x, \frac{2}{3}\right)$ in quadrant II
   c) $\left(-\frac{7}{8}, y\right)$ in quadrant III
   d) $\left(x, -\frac{5}{7}\right)$ in quadrant IV
   e) $\left(x, \frac{1}{3}\right)$, where $x < 0$
   f) $\left(\frac{12}{13}, y\right)$, not in quadrant I

4. If P(θ) is the point at the intersection of the terminal arm of angle θ and the unit circle, determine the exact coordinates of each of the following.
   a) $P(\pi)$
   b) $P\left(-\frac{\pi}{2}\right)$
   c) $P\left(\frac{\pi}{3}\right)$
   d) $P\left(-\frac{\pi}{6}\right)$
   e) $P\left(\frac{3\pi}{4}\right)$
   f) $P\left(-\frac{7\pi}{4}\right)$
   g) $P(4\pi)$
   h) $P\left(\frac{5\pi}{2}\right)$
   i) $P\left(\frac{5\pi}{6}\right)$
   j) $P\left(-\frac{4\pi}{3}\right)$

5. Identify a measure for the central angle θ in the interval $0 \leq \theta < 2\pi$ such that P(θ) is the given point.
   a) $(0, -1)$
   b) $(1, 0)$
   c) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
   d) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
   e) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
   f) $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
   g) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
   h) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
   i) $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
   j) $(-1, 0)$

6. Determine one positive and one negative measure for θ if $P(\theta) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Apply

7. Draw a diagram of the unit circle.
   a) Mark two points, P(θ) and P(θ + π), on your diagram. Use measurements to show that these points have the same coordinates except for their signs.
   b) Choose a different quadrant for the original point, P(θ). Mark it and P(θ + π) on your diagram. Is the result from part a) still true?

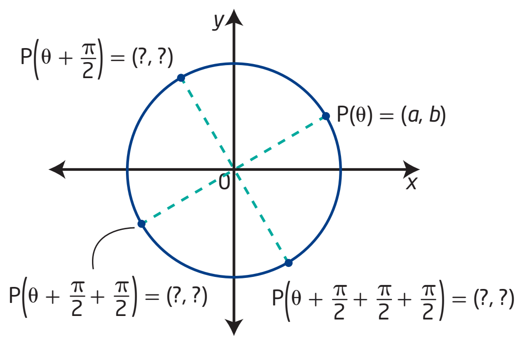
8. **MINI LAB** Determine the pattern in the coordinates of points that are $\frac{1}{4}$ rotation apart on the unit circle.

Step 1 Start with the points $P(0) = (1, 0)$, $P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Show these points on a diagram.

Step 2 Move $+\frac{1}{4}$ rotation from each point. Determine each new point and its coordinates. Show these points on your diagram from step 1.

Step 3 Move $-\frac{1}{4}$ rotation from each original point. Determine each new point and its coordinates. Mark these points on your diagram.

Step 4 How do the values of the *x*-coordinates and *y*-coordinates of points change with each quarter-rotation? Make a copy of the diagram and complete the coordinates to summarize your findings.



9. Use the diagram below to help answer these questions.

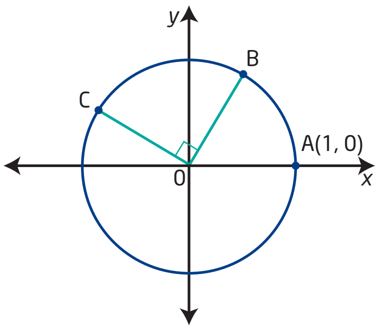
a) What is the equation of this circle?

b) If the coordinates of C are $\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$, what are the coordinates of B?

c) If the measure of $\overset{\frown}{AB}$ is θ, what is an expression for the measure of $\overset{\frown}{AC}$? Note: $\overset{\frown}{AB}$ means the arc length from A to B.

d) Let $P(\theta) = B$. In which quadrant is $P\left(\theta - \frac{\pi}{2}\right)$?

e) What are the maximum and minimum values for either the *x*-coordinates or *y*-coordinates of points on the unit circle?



10. Mya claims that every value of *x* between 0 and 1 can be used to find the coordinates of a point on the unit circle in quadrant I.

a) Do you agree with Mya? Explain.

b) Mya showed the following work to find the *y*-coordinate when $x = 0.807$.

$y = 1 - (0.807)^2$
$= 0.348\ 751$

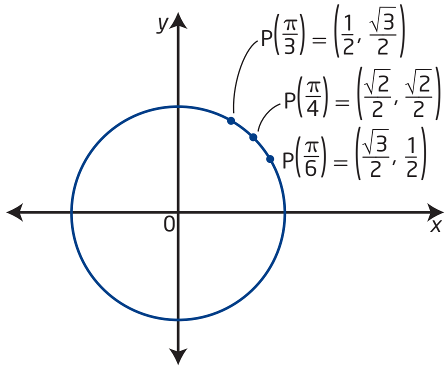
The point on the unit circle is (0.807, 0.348 751).

How can you check Mya's answer? Is she correct? If not, what is the correct answer?

c) If $y = 0.2571$, determine *x* so the point is on the unit circle and in the first quadrant.

11. Wesley enjoys tricks and puzzles. One of his favourite tricks involves remembering the coordinates for $P\left(\frac{\pi}{3}\right)$, $P\left(\frac{\pi}{4}\right)$, and $P\left(\frac{\pi}{6}\right)$. He will not tell you his trick. However, you can discover it for yourself.

a) Examine the coordinates shown on the diagram.



b) What do you notice about the denominators?

c) What do you notice about the numerators of the *x*-coordinates? Compare them with the numerators of the *y*-coordinates. Why do these patterns make sense?

d) Why are square roots involved?

e) Explain this memory trick to a partner.

12. a) Explain, with reference to the unit circle, what the interval $-2\pi \le \theta < 4\pi$ represents.

b) Use your explanation to determine all values for θ in the interval $-2\pi \le \theta < 4\pi$ such that $P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

c) How do your answers relate to the word "coterminal"?

13. If $P(\theta) = \left(-\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$, determine the following.

a) What does $P(\theta)$ represent? Explain using a diagram.

b) In which quadrant does θ terminate?

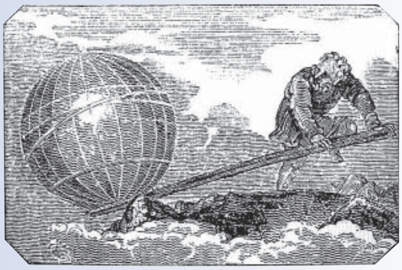
c) Determine the coordinates of $P\left(\theta + \frac{\pi}{2}\right)$.

d) Determine the coordinates of $P\left(\theta - \frac{\pi}{2}\right)$.

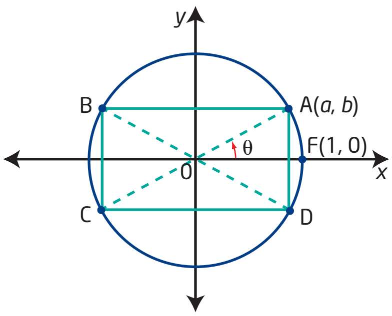
14. In ancient times, determining the perimeter and area of a circle were considered major mathematical challenges. One of Archimedes' greatest contributions to mathematics was his method for approximating π. Now, it is your turn to be a mathematician. Using a unit circle diagram, show the difference between π units and π square units.

Did You Know?

Archimedes was a Greek mathematician, physicist, inventor, and astronomer who lived from 287 BCE–212 BCE. He died in the Roman siege of Syracuse. He is considered one of the greatest mathematicians of all time. He correctly determined the value of π as being between $\frac{22}{7}$ and $\frac{223}{71}$ and proved the area of a circle to be πr^2, where r is the radius.



15. a) In the diagram, A has coordinates (a, b). ABCD is a rectangle with sides parallel to the axes. What are the coordinates of B, C, and D?



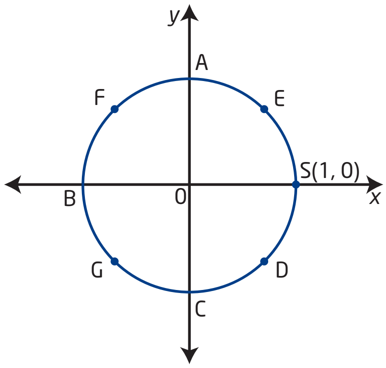
b) $\angle FOA = \theta$, and A, B, C, and D lie on the unit circle. Through which point will the terminal arm pass for each angle? Assume all angles are in standard position.

i) $\theta + \pi$ **ii)** $\theta - \pi$

iii) $-\theta + \pi$ **iv)** $-\theta - \pi$

c) How are the answers in part b) different if θ is given as the measure of arc FA?

16. Use the unit circle diagram to answer the following questions. Points E, F, G, and D are midway between the axes.



a) What angle of rotation creates arc SG? What is the arc length of SG?

b) Which letter on the diagram corresponds to $P\left(\frac{13\pi}{2}\right)$? Explain your answer fully so someone not taking this course would understand. Use a diagram and a written explanation.

c) Between which two points would you find P(5)? Explain.

Extend

17. a) Determine the coordinates of all points where the line represented by $y = -3x$ intersects the unit circle. Give your answers as exact values in simplest form.

b) If one of the points is labelled $P(\theta + \pi)$, draw a diagram and show at least two values for θ. Explain what θ represents.

18. a) $P(\theta)$ lies at the intersection of the unit circle and the line joining A(5, 2) to the origin. Use your knowledge of similar triangles and the unit circle to determine the exact coordinates of $P(\theta)$.

b) Determine the radius of a larger circle with centre at the origin and passing through point A.

c) Write the equation for this larger circle.

19. In previous grades, you used sine and cosine as trigonometric ratios of sides of right triangles. Show how that use of trigonometry relates to the unit circle. Specifically, show that the coordinates of P(θ) can be represented by (cos θ, sin θ) for any θ in the unit circle.

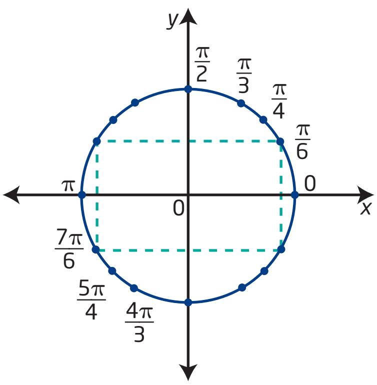
20. You can locate a point in a plane using Cartesian coordinates (x, y), where $|x|$ is the distance from the y-axis and $|y|$ is the distance from the x-axis. You can also locate a point in a plane using (r, θ), where r, $r \geq 0$, is the distance from the origin and θ is the angle of rotation from the positive x-axis. These are known as polar coordinates. Determine the polar coordinates for each point.

a) $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ **b)** $\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{3}\right)$

c) $(2, 2)$ **d)** $(4, -3)$

Create Connections

C1 The diagram represents the unit circle with some positive arc lengths shown.



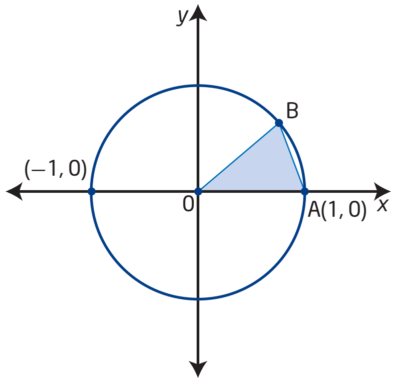
a) Draw a similar diagram in your notebook. Complete the labelling for positive measures.

b) Write the corresponding negative value beside each positive value. Complete this process over the interval $-2\pi \leq \theta < 0$.

c) Give the exact coordinates for the vertices of the dashed rectangle.

d) Identify several patterns from your unit circle diagrams. Patterns can relate to arc lengths, coordinates of points, and symmetry.

C2 Consider the isosceles △AOB drawn in the unit circle.



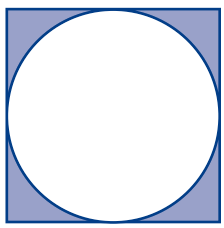
a) If the measure of one of the equal angles is twice the measure of the third angle, determine the exact measure of arc AB.

b) Draw a new △COA in which $P(C) = P\left(B + \dfrac{\pi}{2}\right)$. What is the exact measure of ∠CAO, in radians?

C3 **a)** Draw a diagram of a circle with centre at the origin and radius r units. What is the equation of this circle?

b) Show that the equation of any circle with centre (h, k) and radius r can be expressed as $(x - h)^2 + (y - k)^2 = r^2$. Hint: Use transformations to help with your explanation.

C4 The largest possible unit circle is cut from a square piece of paper as shown.



a) What percent of the paper is cut off? Give your answer to one decimal place.

b) What is the ratio of the circumference of the circle to the perimeter of the original piece of paper?

4.3

Trigonometric Ratios

Focus on…

- relating the trigonometric ratios to the coordinates of points on the unit circle
- determining exact and approximate values for trigonometric ratios
- identifying the measures of angles that generate specific trigonometric values
- solving problems using trigonometric ratios

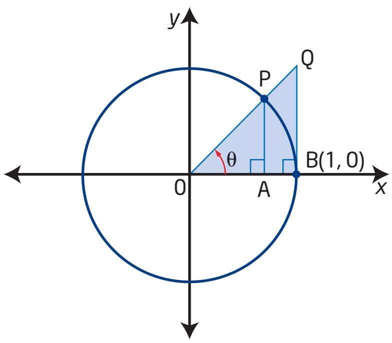
What do a software designer, a civil engineer, an airline pilot, and a long-distance swimmer's support team have in common? All of them use angles and trigonometric ratios to help solve problems. The software designer uses trigonometry to present a 3-D world on a 2-D screen. The engineer uses trigonometric ratios in designs of on-ramps and off-ramps at highway interchanges. A pilot uses an approach angle that is determined based on the tangent ratio. The support team for a long-distance swimmer uses trigonometry to compensate for the effect of wind and currents and to guide the swimmer's direction.

Investigate Trigonometric Ratios and the Unit Circle

Materials

- grid paper
- straight edge
- compass

1. Draw a unit circle as shown, with a positive angle θ in standard position. Work with a partner to describe the location of points P and Q. Be specific.



2. From your drawing, identify a single line segment whose length is equivalent to sin θ. Hint: Use the ratio definition of sin θ and the unit circle to help you.
3. Identify a line segment whose length is equivalent to cos θ and a line segment whose length is equivalent to tan θ in your diagram.
4. From your answers in steps 2 and 3, what could you use to represent the coordinates of point P?

Reflect and Respond

5. Present an argument or proof that the line segment you selected in step 3 for cos θ is correct.

6. What equation relates the coordinates of point P? Does this apply to any point P that lies at the intersection of the terminal arm for an angle θ and the unit circle? Why?

7. What are the maximum and minimum values for cos θ and sin θ? Express your answer in words and using an inequality. Confirm your answer using a calculator.

8. The value of tan θ changes from 0 to undefined for positive values of θ less than or equal to 90°. Explain how this change occurs with reference to angle θ in quadrant I of the unit circle. What happens on a calculator when tan θ is undefined?

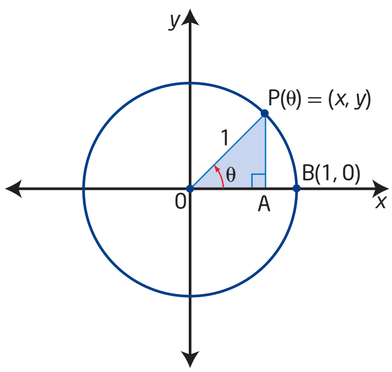
Link the Ideas

Coordinates in Terms of Primary Trigonometric Ratios

If $P(\theta) = (x, y)$ is the point on the terminal arm of angle θ that intersects the unit circle, notice that

- $\cos \theta = \frac{x}{1} = x$, which is the first coordinate of P(θ)
- $\sin \theta = \frac{y}{1} = y$, which is the second coordinate of P(θ)

How do these ratios connect to the right-triangle definition for cosine and sine?



You can describe the coordinates of any point P(θ) as (cos θ, sin θ). This is true for any point P(θ) at the intersection of the terminal arm of an angle θ and the unit circle.

Also, you know that $\tan \theta = \frac{y}{x}$.

Explain how this statement is consistent with the right-triangle definition of the tangent ratio.

Reciprocal Trigonometric Ratios

Three other trigonometric ratios are defined: they are the reciprocals of sine, cosine, and tangent. These are **cosecant**, **secant**, and **cotangent**.

By definition, $\csc \theta = \frac{1}{\sin \theta}$, $\sec \theta = \frac{1}{\cos \theta}$, and $\cot \theta = \frac{1}{\tan \theta}$.

cosecant ratio
- the reciprocal of the sine ratio
- abbreviated csc
- for P(θ) = (x, y) on the unit circle, $\csc \theta = \frac{1}{y}$
- if $\sin \theta = -\frac{\sqrt{3}}{2}$, then $\csc \theta = -\frac{2}{\sqrt{3}}$ or $-\frac{2\sqrt{3}}{3}$

secant ratio
- the reciprocal of the cosine ratio
- abbreviated sec
- for P(θ) = (x, y) on the unit circle, $\sec \theta = \frac{1}{x}$
- if $\cos \theta = \frac{1}{2}$, then $\sec \theta = \frac{2}{1}$ or 2

cotangent ratio
- the reciprocal of the tangent ratio
- abbreviated cot
- for P(θ) = (x, y) on the unit circle, $\cot \theta = \frac{x}{y}$
- if $\tan \theta = 0$, then $\cot \theta$ is undefined

Example 1

Determine the Trigonometric Ratios for Angles in the Unit Circle

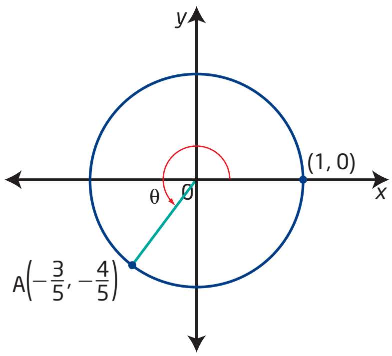
The point $A\left(-\frac{3}{5}, -\frac{4}{5}\right)$ lies at the intersection of the unit circle and the terminal arm of an angle θ in standard position.

a) Draw a diagram to model the situation.

b) Determine the values of the six trigonometric ratios for θ. Express answers in lowest terms.

Solution

a)



b) $\sin \theta = -\frac{4}{5}$ — The y-coordinate of P(θ) is defined as sin θ.

$\cos \theta = -\frac{3}{5}$ — Why is this true?

$\tan \theta = \frac{y}{x}$

$= \frac{-\frac{4}{5}}{-\frac{3}{5}}$ — Explain the arithmetic used to simplify this double fraction.

$= \frac{4}{3}$ — Why does it make sense for tan θ to be positive?

$\csc \theta = \frac{1}{\sin \theta}$

$= -\frac{5}{4}$ — Explain how this answer was determined.

$\sec \theta = \frac{1}{\cos \theta}$ — Read as "sec θ equals the reciprocal of cos θ."

$= -\frac{5}{3}$

$\cot \theta = \frac{1}{\tan \theta}$

$= \frac{3}{4}$

Your Turn

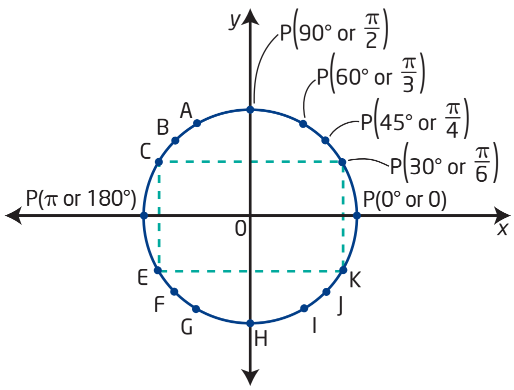
The point $B\left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$ lies at the intersection of the unit circle and the terminal arm of an angle θ in standard position.

a) Draw a diagram to model the situation.

b) Determine the values of the six trigonometric ratios for θ. Express your answers in lowest terms.

Exact Values of Trigonometric Ratios

Exact values for the trigonometric ratios can be determined using special triangles (30°-60°-90° or 45°-45°-90°) and multiples of $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$ or $\theta = 0°, 30°, 45°, 60°$, and $90°$ for points $P(\theta)$ on the unit circle.



How are P(30°), C, E, and K related?

What points have the same coordinates as $P\left(\frac{\pi}{3}\right)$ except for their signs?

For P(45°), what are the coordinates and in which quadrant is θ?

Which special triangle would you use and where would it be placed for $\theta = 135°$?

Example 2

Exact Values for Trigonometric Ratios

Determine the exact value for each. Draw diagrams to illustrate your answers.

a) $\cos \frac{5\pi}{6}$

b) $\sin \left(-\frac{4\pi}{3}\right)$

c) sec 315°

d) cot 270°

Solution

a) The point $P\left(\frac{5\pi}{6}\right)$ lies in quadrant II.

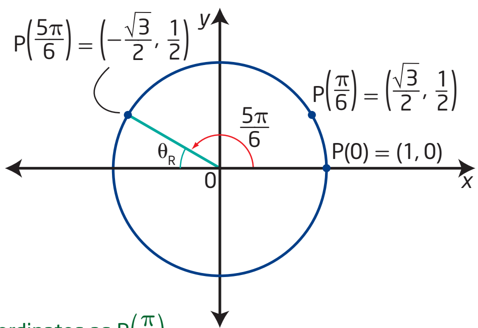
The reference angle for $\frac{5\pi}{6}$ is

$\theta_R = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$.

Its x-coordinate is negative and its y-coordinate is positive.

$$P(\theta) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$



$P\left(\frac{5\pi}{6}\right)$ has the same coordinates as $P\left(\frac{\pi}{6}\right)$, except the x-coordinates have different signs.

Recall that the reference angle, θ_R, is the acute angle formed between the terminal arm and the x-axis.

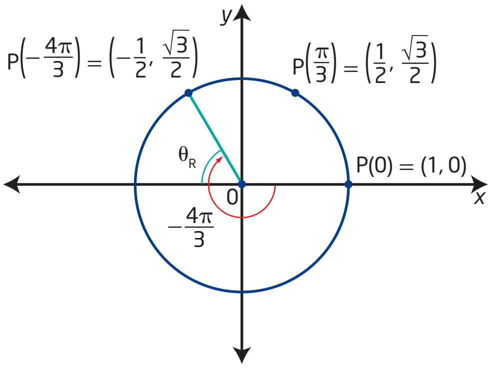
b) $-\frac{4\pi}{3}$ is a clockwise rotation from the positive x-axis.

$P\left(-\frac{4\pi}{3}\right)$ lies in quadrant II.

The reference angle for $-\frac{4\pi}{3}$ is $\theta_R = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

$$P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\sin\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$$



What is a positive coterminal angle for $-\frac{4\pi}{3}$?

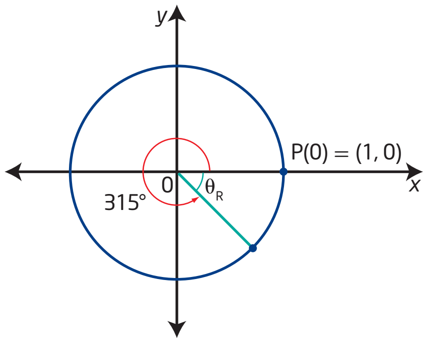
c) An angle of 315° is a counterclockwise rotation that terminates in quadrant IV.

The reference angle for 315° is $\theta_R = 360° - 315° = 45°$.

$P(\theta) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Explain how to get the coordinates for $P(\theta)$.

$$\sec 315° = \frac{1}{\cos 315°}$$
$$= \frac{\sqrt{2}}{1} \text{ or } \sqrt{2}$$



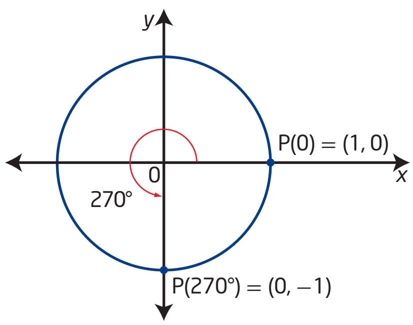
d) An angle of 270° terminates on the negative y-axis.

$P(270°) = (0, -1)$

Since $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$.

Therefore,

$$\cot 270° = \frac{0}{-1}$$
$$= 0$$



Your Turn

Draw diagrams to help you determine the exact value of each trigonometric ratio.

a) $\tan\frac{\pi}{2}$

b) $\csc\frac{7\pi}{6}$

c) $\sin(-300°)$

d) $\sec 60°$

Approximate Values of Trigonometric Ratios

You can determine approximate values for sine, cosine, and tangent using a scientific or graphing calculator. Most calculators can determine trigonometric values for angles measured in degrees or radians. You will need to set the mode to the correct angle measure. Check using

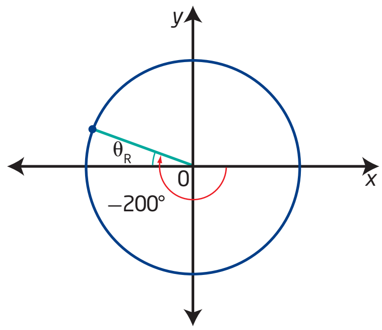
$\cos 60° = 0.5$ (degree mode)
$\cos 60 = -0.952\ 412\ 980...$ (radian mode)

In which quadrant does an angle of 60 terminate?

Most calculators can compute trigonometric ratios for negative angles. However, you should use your knowledge of reference angles and the signs of trigonometric ratios for the quadrant to check that your calculator display is reasonable.

$\cos (-200°) = -0.939\ 692\ 620...$

Is the negative value appropriate? What is the reference angle for −200°? What other trigonometric ratio could you compute as a check?



You can find the value of a trigonometric ratio for cosecant, secant, or cotangent using the correct reciprocal relationship.

$$\begin{aligned}\sec 3.3 &= \frac{1}{\cos 3.3}\\ &= -1.012\ 678\ 973...\\ &\approx -1.0127\end{aligned}$$

Example 3

Approximate Values for Trigonometric Ratios

Determine the approximate value for each trigonometric ratio. Give your answers to four decimal places.

a) $\tan \frac{7\pi}{5}$
b) $\cos 260°$
c) $\sin 4.2$
d) $\csc (-70°)$

Solution

a) $\frac{7\pi}{5}$ is measured in radians.

$$\begin{aligned}\tan \frac{7\pi}{5} &= 3.077\ 683\ 537...\\ &\approx 3.0777\end{aligned}$$

In which quadrant does an angle of $\frac{7\pi}{5}$ terminate?

Make sure your calculator is in radian mode.

Why is the answer positive?

b) $$\begin{aligned}\cos 260° &= -0.173\ 648\ 177...\\ &\approx -0.1736\end{aligned}$$

In which quadrant does 260° terminate?

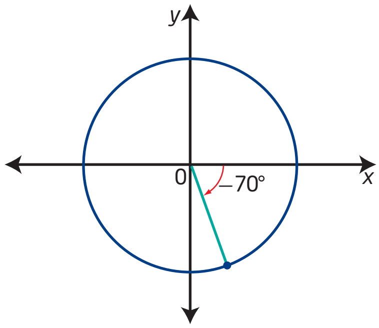
c) $\sin 4.2 = -0.871\ 575\ 772\ldots$
≈ -0.8716

Which angle mode do you need here?
Why is the answer negative?

d) An angle of $-70°$ terminates in quadrant IV.
The y-coordinate for points in quadrant IV is negative.

$$\csc(-70°) = \frac{1}{\sin(-70°)}$$
$$= -1.064\ 177\ 772\ldots$$
$$\approx -1.0642$$

What steps are needed to evaluate $\frac{1}{\sin(-70°)}$ on your calculator?



Your Turn

What is the approximate value for each trigonometric ratio? Round answers to four decimal places. Justify the sign of each answer.

a) $\sin 1.92$
b) $\tan(-500°)$
c) $\sec 85.4°$
d) $\cot 3$

Approximate Values of Angles

How can you find the measure of an angle when the value of the trigonometric ratio is given? To reverse the process (for example, to determine θ if you know $\sin\theta$), use the inverse trigonometric function keys on a calculator.

$\sin 30° = 0.5 \Longrightarrow \sin^{-1} 0.5 = 30°$

Note that $\sin^{-1}$ is an abbreviation for "the inverse of sine." Do not confuse this with $(\sin 30°)^{-1}$, which means $\frac{1}{\sin 30°}$, or the reciprocal of $\sin 30°$.

The calculator keys $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ return one answer only, when there are often two angles with the same trigonometric function value in any full rotation. In general, it is best to use the reference angle applied to the appropriate quadrants containing the terminal arm of the angle.

Example 4

Find Angles Given Their Trigonometric Ratios

Determine the measures of all angles that satisfy the following. Use diagrams in your explanation.

a) $\sin \theta = 0.879$ in the domain $0 \leq \theta < 2\pi$. Give answers to the nearest tenth of a radian.

b) $\cos \theta = -0.366$ in the domain $0° \leq \theta < 360°$. Give answers to the nearest tenth of a degree.

c) $\tan \theta = \sqrt{3}$ in the domain $-180° \leq \theta < 180°$. Give exact answers.

d) $\sec \theta = \dfrac{2}{\sqrt{3}}$ in the domain $-2\pi \leq \theta < 2\pi$. Give exact answers.

Solution

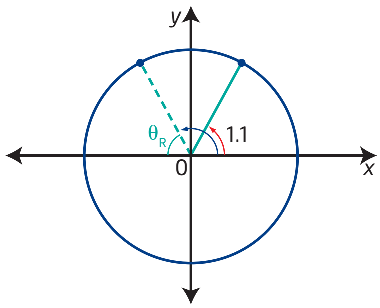
Did You Know?

By convention, if the domain is given in radian measure, express answers in radians. If the domain is expressed using degrees, give the answers in degrees.

a) $\sin \theta > 0$ in quadrants I and II.
The domain consists of one positive rotation.
Therefore, two answers need to be identified.

$$\sin^{-1} 0.879 = 1.073\ 760\ 909\ldots$$
$$\approx 1.1$$

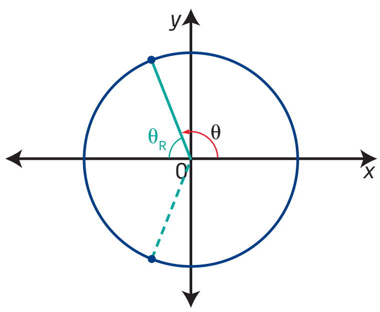
Use a calculator in radian mode.



In quadrant I, $\theta \approx 1.1$, to the nearest tenth. This is the reference angle.
In quadrant II, $\theta \approx \pi - 1.1$ or 2.0, to the nearest tenth.
The answers, to the nearest tenth of a radian, are 1.1 and 2.0.

b) $\cos \theta < 0$ in quadrants II and III.

Why will the answer be measured in degrees?



$\cos^{-1} (-0.366) \approx 111.5°$, to the nearest tenth.
This answer is in quadrant II.
The reference angle for other answers is 68.5°.
In quadrant III, $\theta \approx 180° + 68.5°$ or 248.5°.

Did you check that your calculator is in degree mode?

How do you determine this reference angle from 111.5°?

The answers, to the nearest tenth of a degree, are 111.5° and 248.5°.

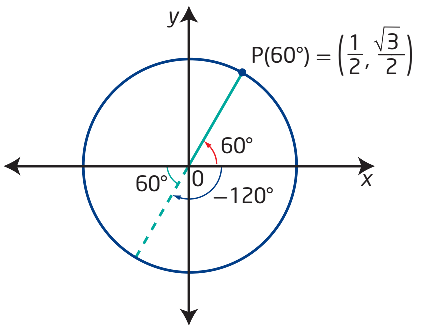
c) $\tan \theta > 0$ in quadrants I and III.

The domain includes both quadrants. In the positive direction an answer will be in quadrant I, and in the negative direction an answer will be in quadrant III.

To answer with exact values, work with the special coordinates on a unit circle.

$\tan 60° = \sqrt{3}$

How do you know that $\tan 60° = \sqrt{3}$? Could you use a calculator here?

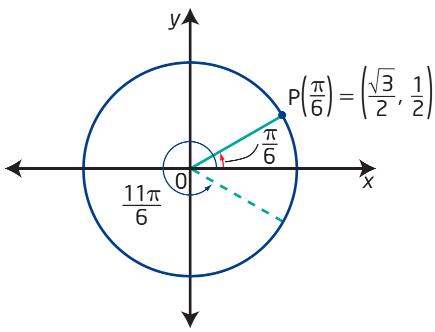


In quadrant I, from the domain $0° \leq \theta < 180°$, $\theta = 60°$. This is the reference angle. In quadrant III, from the domain $-180° \leq \theta < 0°$, $\theta = -180° + 60°$ or $-120°$.

The exact answers are 60° and −120°.

d) $\sec \theta > 0$ in quadrants I and IV since $\sec \theta = \frac{1}{\cos \theta}$ and $\cos \theta > 0$ in quadrants I and IV.

The domain includes four quadrants in both the positive and negative directions. Thus, there are two positive answers and two negative answers.



$\cos \theta = \frac{1}{\sec \theta} = \frac{\sqrt{3}}{2}$

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6}$ and $\theta = \frac{11\pi}{6}$ in the domain $0 \leq \theta < 2\pi$.

$\theta = -\frac{\pi}{6}$ and $\theta = -\frac{11\pi}{6}$ in the domain $-2\pi \leq \theta < 0$.

How do coterminal angles help?

The exact answers in radians are $\frac{\pi}{6}$, $\frac{11\pi}{6}$, $-\frac{\pi}{6}$, and $-\frac{11\pi}{6}$.

Your Turn

Determine the measures of all angles that satisfy each of the following. Use diagrams to show the possible answers.

a) $\cos \theta = 0.843$ in the domain $-360° < \theta < 180°$. Give approximate answers to the nearest tenth.

b) $\sin \theta = 0$ in the domain $0° \leq \theta \leq 180°$. Give exact answers.

c) $\cot \theta = -2.777$ in the domain $-\pi \leq \theta \leq \pi$. Give approximate answers to the nearest tenth.

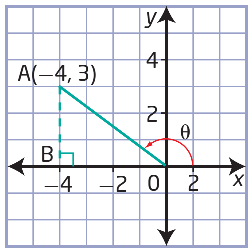
d) $\csc \theta = -\frac{2}{\sqrt{2}}$ in the domain $-2\pi \leq \theta \leq \pi$. Give exact answers.

Example 5

Calculating Trigonometric Values for Points Not on the Unit Circle

The point A(−4, 3) lies on the terminal arm of an angle θ in standard position. What is the exact value of each trigonometric ratio for θ?

Solution



Confirm this using the Pythagorean theorem.

△ABO is a right triangle.

Identify trigonometric values for θ using the lengths of the sides of △ABO.

△ABO has sides of lengths 3, 4, and 5.

Recall that OA is a length and the segments OB and BA are considered as directed lengths.

$$\sin \theta = \frac{y}{r} = \frac{3}{5}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{5}{3}$$

$$\cos \theta = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5}$$

$$\sec \theta = \frac{1}{\cos \theta} = -\frac{5}{4}$$

$$\tan \theta = \frac{y}{x} = \frac{3}{-4} = -\frac{3}{4}$$

$$\cot \theta = \frac{1}{\tan \theta} = -\frac{4}{3}$$

Your Turn

The point D(−5, −12) lies on the terminal arm of an angle θ in standard position. What is the exact value of each trigonometric ratio for θ?

Key Ideas

- Points that are on the intersection of the terminal arm of an angle θ in standard position and the unit circle can be defined using trigonometric ratios.

  $P(\theta) = (\cos\theta, \sin\theta)$

- Each primary trigonometric ratio—sine, cosine, and tangent—has a reciprocal trigonometric ratio. The reciprocals are cosecant, secant, and cotangent, respectively.

  $\csc\theta = \dfrac{1}{\sin\theta}$ $\quad\sec\theta = \dfrac{1}{\cos\theta}$ $\quad\cot\theta = \dfrac{1}{\tan\theta}$

  If $\sin\theta = \frac{2}{3}$, then $\csc\theta = \frac{3}{2}$, and vice versa.

- You can determine the trigonometric ratios for any angle in standard position using the coordinates of the point where the terminal arm intersects the unit circle.
- Exact values of trigonometric rations for special angles such as $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$ and their multiples may be determined using the coordinates of points on the unit circle.
- You can determine approximate values for trigonometric ratios using a calculator in the appropriate mode: radians or degrees.
- You can use a scientific or graphing calculator to determine an angle measure given the value of a trigonometric ratio. Then, use your knowledge of reference angles, coterminal angles, and signs of ratios in each quadrant to determine other possible angle measures. Unless the domain is restricted, there are an infinite number of answers.
- Determine the trigonometric ratios for an angle θ in standard position from the coordinates of a point on the terminal arm of θ and right triangle definitions of the trigonometric ratios.

Check Your Understanding

Practise

1. What is the exact value for each trigonometric ratio?

a) $\sin 45°$ **b)** $\tan 30°$

c) $\cos \frac{3\pi}{4}$ **d)** $\cot \frac{7\pi}{6}$

e) $\csc 210°$ **f)** $\sec (-240°)$

g) $\tan \frac{3\pi}{2}$ **h)** $\sec \pi$

i) $\cot (-120°)$ **j)** $\cos 390°$

k) $\sin \frac{5\pi}{3}$ **l)** $\csc 495°$

2. Determine the approximate value for each trigonometric ratio. Give answers to two decimal places.

a) $\cos 47°$ **b)** $\cot 160°$

c) $\sec 15°$ **d)** $\csc 4.71$

e) $\sin 5$ **f)** $\tan 0.94$

g) $\sin \frac{5\pi}{7}$ **h)** $\tan 6.9$

i) $\cos 302°$ **j)** $\sin \left(-\frac{11\pi}{19}\right)$

k) $\cot 6$ **l)** $\sec (-270°)$

3. If θ is an angle in standard position with the following conditions, in which quadrants may θ terminate?

a) $\cos \theta > 0$

b) $\tan \theta < 0$

c) $\sin \theta < 0$

d) $\sin \theta > 0$ and $\cot \theta < 0$

e) $\cos \theta < 0$ and $\csc \theta > 0$

f) $\sec \theta > 0$ and $\tan \theta > 0$

4. Express the given quantity using the same trigonometric ratio and its reference angle. For example, $\cos 110° = -\cos 70°$. For angle measures in radians, give exact answers. For example, $\cos 3 = -\cos (\pi - 3)$.

a) $\sin 250°$

b) $\tan 290°$

c) $\sec 135°$

d) $\cos 4$

e) $\csc 3$

f) $\cot 4.95$

5. For each point, sketch two coterminal angles in standard position whose terminal arm contains the point. Give one positive and one negative angle, in radians, where neither angle exceeds one full rotation.

a) $(3, 5)$

b) $(-2, -1)$

c) $(-3, 2)$

d) $(5, -2)$

6. Indicate whether each trigonometric ratio is positive or negative. Do not use a calculator.

a) $\cos 300°$

b) $\sin 4$

c) $\cot 156°$

d) $\csc (-235°)$

e) $\tan \frac{13\pi}{6}$

f) $\sec \frac{17\pi}{3}$

7. Determine each value. Explain what the answer means.

a) $\sin^{-1} 0.2$

b) $\tan^{-1} 7$

c) $\sec 450°$

d) $\cot (-180°)$

8. The point $P(\theta) = \left(\frac{3}{5}, y\right)$ lies on the terminal arm of an angle θ in standard position and on the unit circle. $P(\theta)$ is in quadrant IV.

a) Determine y.

b) What is the value of $\tan \theta$?

c) What is the value of $\csc \theta$?

Apply

9. Determine the exact value of each expression.

a) $\cos 60° + \sin 30°$

b) $(\sec 45°)^2$

c) $\left(\cos \frac{5\pi}{3}\right)\left(\sec \frac{5\pi}{3}\right)$

d) $(\tan 60°)^2 - (\sec 60°)^2$

e) $\left(\cos \frac{7\pi}{4}\right)^2 + \left(\sin \frac{7\pi}{4}\right)^2$

f) $\left(\cot \frac{5\pi}{6}\right)^2$

10. Determine the exact measure of all angles that satisfy the following. Draw a diagram for each.

a) $\sin \theta = -\frac{1}{2}$ in the domain $0 \le \theta < 2\pi$

b) $\cot \theta = 1$ in the domain $-\pi \le \theta < 2\pi$

c) $\sec \theta = 2$ in the domain $-180° \le \theta < 90°$

d) $(\cos \theta)^2 = 1$ in the domain $-360° \le \theta < 360°$

11. Determine the approximate measure of all angles that satisfy the following. Give answers to two decimal places. Use diagrams to show the possible answers.

a) $\cos \theta = 0.42$ in the domain $-\pi \le \theta \le \pi$

b) $\tan \theta = -4.87$ in the domain $-\frac{\pi}{2} \le \theta \le \pi$

c) $\csc \theta = 4.87$ in the domain $-360° \le \theta < 180°$

d) $\cot \theta = 1.5$ in the domain $-180° \le \theta < 360°$

12. Determine the exact values of the other five trigonometric ratios under the given conditions.

a) $\sin \theta = \frac{3}{5}$, $\frac{\pi}{2} < \theta < \pi$

b) $\cos \theta = \frac{-2\sqrt{2}}{3}$, $-\pi \le \theta \le \frac{3\pi}{2}$

c) $\tan \theta = \frac{2}{3}$, $-360° < \theta < 180°$

d) $\sec \theta = \frac{4\sqrt{3}}{3}$, $-180° \le \theta \le 180°$

13. Using the point B(−2, −3), explain how to determine the exact value of cos θ given that B is a point on the terminal arm of an angle θ in standard position.

14. The measure of angle θ in standard position is 4900°.

a) Describe θ in terms of revolutions. Be specific.

b) In which quadrant does 4900° terminate?

c) What is the measure of the reference angle?

d) Give the value of each trigonometric ratio for 4900°.

15. a) Determine the positive value of $\sin(\cos^{-1} 0.6)$. Use your knowledge of the unit circle to explain why the answer is a rational number.

b) Without calculating, what is the positive value of $\cos(\sin^{-1} 0.6)$? Explain.

16. a) Jason got an answer of 1.051 176 209 when he used a calculator to determine the value of $\sec \frac{40\pi}{7}$. Is he correct? If not, where did he make his mistake?

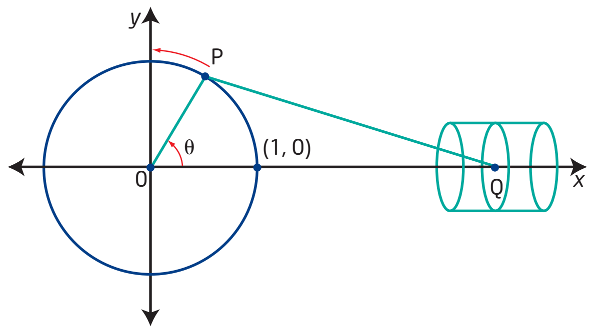
b) Describe the steps you would use to determine an approximate value for $\sec \frac{40\pi}{7}$ on your calculator.

17. a) Arrange the following values of sine in increasing order.
sin 1, sin 2, sin 3, sin 4

b) Show what the four values represent on a diagram of the unit circle. Use your diagram to justify the order from part a).

c) Predict the correct increasing order for cos 1, cos 2, cos 3, and cos 4. Check with a calculator. Was your prediction correct?

18. Examine the diagram. A piston rod, PQ, is connected to a wheel at P and to a piston at Q. As P moves around the wheel in a counterclockwise direction, Q slides back and forth.



a) What is the maximum distance that Q can move?

b) If the wheel starts with P at (1, 0) and rotates at 1 radian/s, draw a sketch to show where P will be located after 1 min.

c) What distance will Q have moved 1 s after start-up? Give your answer to the nearest hundredth of a unit.

19. Each point lies on the terminal arm of an angle θ in standard position. Determine θ in the specified domain. Round answers to the nearest hundredth of a unit.

a) A(−3, 4), $0 < \theta \leq 4\pi$

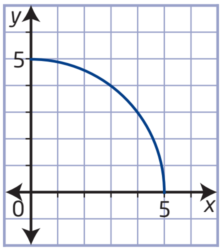
b) B(5, −1), $-360° \leq \theta < 360°$

c) C(−2, −3), $-\frac{3\pi}{2} < \theta < \frac{7\pi}{2}$

Extend

20. Draw △ABC with ∠A = 15° and ∠C = 90°. Let BC = 1. D is a point on AC such that ∠DBC = 60°. Use your diagram to help you show that $\tan 15° = \frac{1}{\sqrt{3} + 2}$.

21. The diagram shows a quarter-circle of radius 5 units. Consider the point on the curve where $x = 2.5$. Show that this point is one-third the distance between (0, 5) and (5, 0) on the arc of the circle.



22. *Alice Through the Looking Glass* by Lewis Carroll introduced strange new worlds where time ran backwards. Your challenge is to imagine a unit circle in which a positive rotation is defined to be clockwise. Assume the coordinate system remains as we know it.

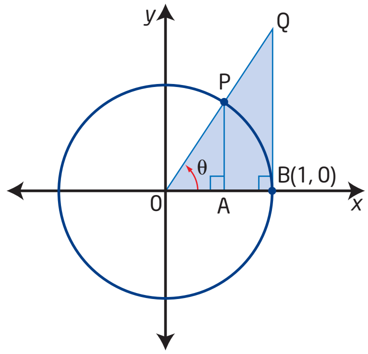
a) Draw a unit circle in which positive angles are measured clockwise from (0, 1). Label where $R\left(\frac{\pi}{6}\right)$, $R\left(\frac{5\pi}{6}\right)$, $R\left(\frac{7\pi}{6}\right)$, and $R\left(\frac{11\pi}{6}\right)$ are on your new unit circle.

b) What are the coordinates for the new $R\left(\frac{\pi}{6}\right)$ and $R\left(\frac{5\pi}{6}\right)$?

c) How do angles in this new system relate to conventional angles in standard position?

d) How does your new system of angle measure relate to bearings in navigation? Explain.

23. In the investigation at the beginning of this section, you identified line segments whose lengths are equivalent to $\cos \theta$, $\sin \theta$, and $\tan \theta$ using the diagram shown.



a) Determine a line segment from the diagram whose length is equivalent to $\sec \theta$. Explain your reasoning

b) Make a copy of the diagram. Draw a horizontal line tangent to the circle that intersects the positive y-axis at C and OQ at D. Now identify segments whose lengths are equivalent to $\csc \theta$ and $\cot \theta$. Explain your reasoning.

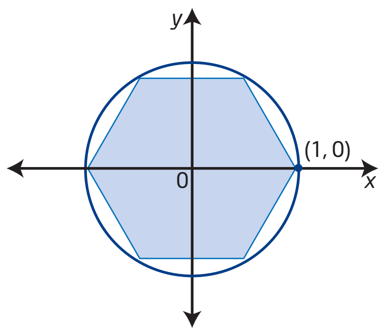
Create Connections

C1 **a)** Paula sees that sine ratios increase from 0 to 1 in quadrant 1. She concludes that the sine relation is increasing in quadrant I. Show whether Paula is correct using specific values for sine.

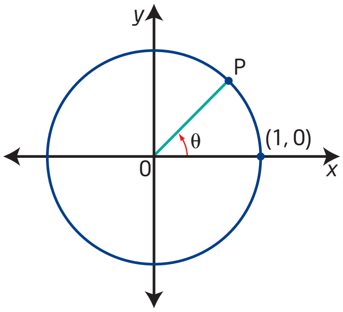
b) Is sine increasing in quadrant II? Explain why or why not.

c) Does the sine ratio increase in any other quadrant, and if so, which? Explain.

C2 A regular hexagon is inscribed in the unit circle as shown. If one vertex is at (1, 0), what are the exact coordinates of the other vertices? Explain your reasoning.



C3 Let P be the point of intersection of the unit circle and the terminal arm of an angle θ in standard position.



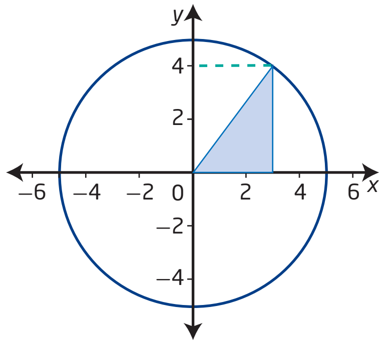
a) What is a formula for the slope of OP? Write your formula in terms of trigonometric ratios.

b) Does your formula apply in every quadrant? Explain.

c) Write an equation for any line OP. Use your trigonometric value for the slope.

d) Use transformations to show that your equation from part c) applies to any line where the slope is defined.

C4 Use the diagram to help find the value of each expression.



a) $\sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

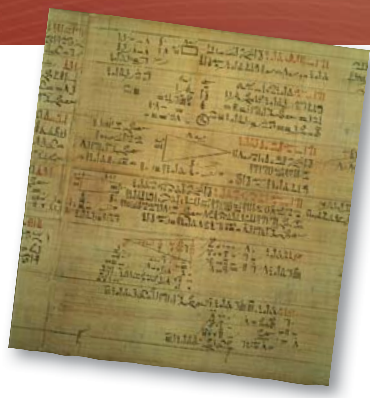
b) $\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

c) $\csc\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$, where the angle is in quadrant II

d) $\sin\left(\tan^{-1}\left(-\frac{4}{3}\right)\right)$, where the angle is in quadrant IV

Project Corner — History of Angle Measurement

- The use of the angular measurement unit "degree" is believed to have originated with the Babylonians. One theory is that their division of a circle into 360 parts is approximately the number of days in a year.
- Degree measures can also be subdivided into minutes (′) and seconds (″), where one degree is divided into 60 min, and one minute is divided into 60 s. For example, $30.1875° = 30°\ 11'\ 15''$.
- The earliest textual evidence of π dates from about 2000 B.C.E., with recorded approximations by the Babylonians $\left(\frac{25}{8}\right)$ and the Egyptians $\left(\frac{256}{81}\right)$. Roger Cotes (1682–1716) is credited with the concept of radian measure of angles, although he did not name the unit.
- The radian is widely accepted as the standard unit of angular measure in many fields of mathematics and in physics. The use of radians allows for the simplification of formulas and provides better approximations.
- What are some alternative units for measuring angles? What are some advantages and disadvantages of these units? What are some contexts in which these units are used?



Rhind Papyrus, ancient Egypt c1650 B.C.E.

4.4

Introduction to Trigonometric Equations

Focus on...

- algebraically solving first-degree and second-degree trigonometric equations in radians and in degrees
- verifying that a specific value is a solution to a trigonometric equation
- identifying exact and approximate solutions of a trigonometric equation in a restricted domain
- determining the general solution of a trigonometric equation

This old silver-gelatin photograph of traditional Kwakiutl spear fishing was taken in 1914 by Edward S. Curtis. The Kwakiutl First Nation's people have lived on the north-eastern shores of Vancouver Island for thousands of years. Today, the band council is based in Fort Rupert and owns 295 hectares of land in the area.

Many situations in nature involve cyclical patterns, such as average daily temperature at a specific location. Other applications of trigonometry relate to electricity or the way light passes from air into water. When you look at a fish in the water, it is not precisely where it appears to be, due to the refraction of light. The Kwakiutl peoples from Northwest British Columbia figured this out centuries ago. They became expert spear fishermen.

In this section, you will explore how to use algebraic techniques, similar to those used in solving linear and quadratic equations, to solve **trigonometric equations**. Your knowledge of coterminal angles, points on the unit circle, and inverse trigonometric functions will be important for understanding the solution of trigonometric equations.

trigonometric equation

- an equation involving trigonometric ratios

Investigate Trigonometric Equations

Did You Know?

In equations, mathematicians often use the notation $\cos^2 \theta$. This means the same as $(\cos \theta)^2$.

1. What are the exact measures of θ if $\cos \theta = -\frac{1}{2}$, $0 \le \theta < 2\pi$? How is the equation related to $2 \cos \theta + 1 = 0$?
2. What is the answer for step 1 if the domain is given as $0° \le \theta < 360°$?
3. What are the approximate measures for θ if $3 \cos \theta + 1 = 0$ and the domain is $0 \le \theta < 2\pi$?

4. Set up a T-chart like the one below. In the left column, show the steps you would use to solve the quadratic equation $x^2 - x = 0$. In the right column, show similar steps that will lead to the solution of the trigonometric equation $\cos^2 \theta - \cos \theta = 0$, $0 \le \theta < 2\pi$.

Quadratic Equation	Trigonometric Equation

Reflect and Respond

5. How is solving the equations in steps 1 to 3 similar to solving a linear equation? How is it different? Use examples.
6. When solving a trigonometric equation, how do you know whether to give your answers in degrees or radians?
7. Identify similarities and differences between solving a quadratic equation and solving a trigonometric equation that is quadratic.

Link the Ideas

In the investigation, you explored solving trigonometric equations. Did you realize that in Section 4.3 you were already solving simple trigonometric equations? The same processes will be used each time you solve a trigonometric equation, and these processes are the same as those used in solving linear and quadratic equations.

The notation $[0, \pi]$ represents the interval from 0 to π inclusive and is another way of writing $0 \le \theta \le \pi$.

- $\theta \in (0, \pi)$ means the same as $0 < \theta < \pi$.
- $\theta \in [0, \pi)$ means the same as $0 \le \theta < \pi$.

How would you show $-\pi < \theta \le 2\pi$ using interval notation?

Example 1

Solve Trigonometric Equations

Solve each trigonometric equation in the specified domain.

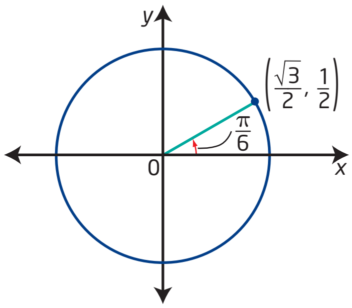
a) $5 \sin \theta + 2 = 1 + 3 \sin \theta$, $0 \le \theta < 2\pi$

b) $3 \csc x - 6 = 0$, $0° \le x < 360°$

Solution

a)

$$\begin{aligned} 5 \sin \theta + 2 &= 1 + 3 \sin \theta \\ 5 \sin \theta + 2 - 3 \sin \theta &= 1 + 3 \sin \theta - 3 \sin \theta \\ 2 \sin \theta + 2 &= 1 \\ 2 \sin \theta + 2 - 2 &= 1 - 2 \\ 2 \sin \theta &= -1 \\ \sin \theta &= -\frac{1}{2} \end{aligned}$$



The reference angle is $\theta_R = \frac{\pi}{6}$.

In which quadrants must θ terminate if $\sin \theta = -\frac{1}{2}$?

$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ (quadrant III)

$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ (quadrant IV)

The solutions are $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$ in the domain $0 \le \theta < 2\pi$.

b) $3 \csc x - 6 = 0$

$3 \csc x = 6$

$\csc x = 2$ What operations were performed to arrive at this equation?

If $\csc x = 2$, then $\sin x = \frac{1}{2}$

$x = 30°$ and $150°$ Explain how to arrive at these answers.

The solutions are $x = 30°$ and $x = 150°$ in the domain $0° \le x < 360°$.

Your Turn

Solve each trigonometric equation in the specified domain.

a) $3 \cos \theta - 1 = \cos \theta + 1, -2\pi \le \theta \le 2\pi$

b) $4 \sec x + 8 = 0, 0° \le x < 360°$

Example 2

Factor to Solve a Trigonometric Equation

Solve for θ.

$\tan^2 \theta - 5 \tan \theta + 4 = 0, 0 \le \theta < 2\pi$

Give solutions as exact values where possible. Otherwise, give approximate angle measures, to the nearest thousandth of a radian.

Solution

$\tan^2 \theta - 5 \tan \theta + 4 = 0$

$(\tan \theta - 1)(\tan \theta - 4) = 0$

How is this similar to solving $x^2 - 5x + 4 = 0$?

$\tan \theta - 1 = 0$ or $\tan \theta - 4 = 0$

$\tan \theta = 1$ $\quad$ $\tan \theta = 4$

In which quadrants is $\tan \theta > 0$?

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ $\quad$ $\tan^{-1} 4 = \theta$

What angle mode must your calculator be in to find $\tan^{-1} 4$?

$\theta = 1.3258...$

$\theta \approx 1.326$ is a measure in quadrant I.

How do you know that 1.326 is in quadrant I?

In quadrant III,

$\theta = \pi + \theta_R$
$= \pi + \tan^{-1} 4$
$= \pi + 1.3258...$
$= 4.467\ 410\ 317...$
≈ 4.467

Why is $\tan^{-1} 4$ used as the reference angle here?

The solutions are $\theta = \frac{\pi}{4}$, $\theta = \frac{5\pi}{4}$ (exact), $\theta \approx 1.326$, and $\theta \approx 4.467$ (to the nearest thousandth).

Your Turn

Solve for θ.

$\cos^2 \theta - \cos \theta - 2 = 0$, $0° \le \theta < 360°$

Give solutions as exact values where possible. Otherwise, give approximate measures to the nearest thousandth of a degree.

Example 3

General Solution of a Trigonometric Equation

a) Solve for x in the interval $0 \le x < 2\pi$ if $\sin^2 x - 1 = 0$. Give answers as exact values.

b) Determine the general solution for $\sin^2 x - 1 = 0$ over the real numbers if x is measured in radians.

Solution

a) Method 1: Use Square Root Principles

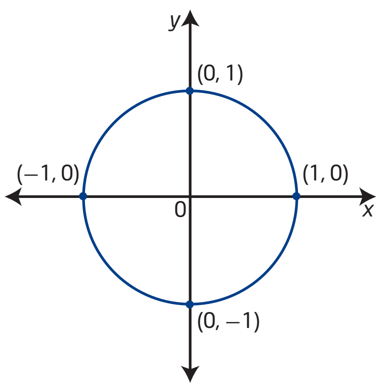
$\sin^2 x - 1 = 0$
$\sin^2 x = 1$
$\sin x = \pm 1$
$\sin x = 1$ or $\sin x = -1$

Why are there two values for $\sin x$?

If $\sin x = 1$, then $x = \frac{\pi}{2}$.

If $\sin x = -1$, then $x = \frac{3\pi}{2}$.

Where did $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ come from? Why is $\frac{5\pi}{2}$ not an acceptable answer?



Method 2: Use Factoring

$$\sin^2 x - 1 = 0$$
$$(\sin x - 1)(\sin x + 1) = 0$$
$$\sin x - 1 = 0 \quad \text{or} \quad \sin x + 1 = 0$$

Continue as in Method 1.

Check: $x = \frac{\pi}{2}$ $\qquad$ $x = \frac{3\pi}{2}$

Left Side	Right Side	Left Side	Right Side
$\sin^2 x - 1$	0	$\sin^2 x - 1$	0
$= \left(\sin \frac{\pi}{2}\right)^2 - 1$		$= \left(\sin \frac{3\pi}{2}\right)^2 - 1$	
$= 1^2 - 1$		$= (-1)^2 - 1$	
$= 0$		$= 0$	

Both answers are verified.

The solution is $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

b) If the domain is real numbers, you can make an infinite number of rotations on the unit circle in both a positive and a negative direction.

Values corresponding to $x = \frac{\pi}{2}$ are $\ldots -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$

What patterns do you see in these values for θ?

Values corresponding to $x = \frac{3\pi}{2}$ are $\ldots -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$

Do you see that the terminal arm is at the point (0, 1) or (0, −1) with any of the angles above?

An expression for the values corresponding to $x = \frac{\pi}{2}$ is $x = \frac{\pi}{2} + 2\pi n$, where $n \in \mathrm{I}$.

An expression for the values corresponding to $x = \frac{3\pi}{2}$ is $x = \frac{3\pi}{2} + 2\pi n$, where $n \in \mathrm{I}$.

The two expressions above can be combined to form the general solution $x = \frac{\pi}{2} + \pi n$, where $n \in \mathrm{I}$.

The solution can also be described as "odd integral multiples of $\frac{\pi}{2}$." In symbols, this is written as $(2n + 1)\left(\frac{\pi}{2}\right)$, $n \in \mathrm{I}$.

How can you show algebraically that $(2n + 1)\left(\frac{\pi}{2}\right)$, $n \in \mathrm{I}$, and $\frac{\pi}{2} + \pi n$, $n \in \mathrm{I}$, are equivalent?

Did You Know?

$2n$, where $n \in \mathrm{I}$, represents all even integers.

$2n + 1$, where $n \in \mathrm{I}$, is an expression for all odd integers.

Your Turn

a) If $\cos^2 x - 1 = 0$, solve for x in the domain $0° \leq x < 360°$. Give solutions as exact values.

b) Determine the general solution for $\cos^2 x - 1 = 0$, where the domain is real numbers measured in degrees.

Key Ideas

- To solve a trigonometric equation algebraically, you can use the same techniques as used in solving linear and quadratic equations.
- When you arrive at $\sin \theta = a$ or $\cos \theta = a$ or $\tan \theta = a$, where $a \in \mathrm{R}$, then use the unit circle for exact values of θ and inverse trigonometric function keys on a calculator for approximate measures. Use reference angles to find solutions in other quadrants.
- To solve a trigonometric equation involving $\csc \theta$, $\sec \theta$, or $\cot \theta$, you may need to work with the related reciprocal value(s).
- To determine a general solution or if the domain is real numbers, find the solutions in one positive rotation (2π or $360°$). Then, use the concept of coterminal angles to write an expression that identifies all possible measures.

Check Your Understanding

Practise

1. Without solving, determine the number of solutions for each trigonometric equation in the specified domain. Explain your reasoning.

a) $\sin \theta = \frac{\sqrt{3}}{2}$, $0 \le \theta < 2\pi$

b) $\cos \theta = \frac{1}{\sqrt{2}}$, $-2\pi \le \theta < 2\pi$

c) $\tan \theta = -1$, $-360° \le \theta \le 180°$

d) $\sec \theta = \frac{2\sqrt{3}}{3}$, $-180° \le \theta < 180°$

2. The equation $\cos \theta = \frac{1}{2}$, $0 \le \theta < 2\pi$, has solutions $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Suppose the domain is not restricted.

a) What is the general solution corresponding to $\theta = \frac{\pi}{3}$?

b) What is the general solution corresponding to $\theta = \frac{5\pi}{3}$?

3. Determine the exact roots for each trigonometric equation or statement in the specified domain.

a) $2 \cos \theta - \sqrt{3} = 0$, $0 \le \theta < 2\pi$

b) $\csc \theta$ is undefined, $0° \le \theta < 360°$

c) $5 - \tan^2 \theta = 4$, $-180° \le \theta \le 360°$

d) $\sec \theta + \sqrt{2} = 0$, $-\pi \le \theta \le \frac{3\pi}{2}$

4. Solve each equation for $0 \le \theta < 2\pi$. Give solutions to the nearest hundredth of a radian.

a) $\tan \theta = 4.36$

b) $\cos \theta = -0.19$

c) $\sin \theta = 0.91$

d) $\cot \theta = 12.3$

e) $\sec \theta = 2.77$

f) $\csc \theta = -1.57$

5. Solve each equation in the specified domain.

a) $3 \cos \theta - 1 = 4 \cos \theta$, $0 \le \theta < 2\pi$

b) $\sqrt{3} \tan \theta + 1 = 0$, $-\pi \le \theta \le 2\pi$

c) $\sqrt{2} \sin x - 1 = 0$, $-360° < x \le 360°$

d) $3 \sin x - 5 = 5 \sin x - 4$, $-360° \le x < 180°$

e) $3 \cot x + 1 = 2 + 4 \cot x$, $-180° < x < 360°$

f) $\sqrt{3} \sec \theta + 2 = 0$, $-\pi \le \theta \le 3\pi$

6. Copy and complete the table to express each domain or interval using the other notation.

	Domain	Interval Notation
a)	$-2\pi \le \theta \le 2\pi$	
b)	$-\frac{\pi}{3} \le \theta \le \frac{7\pi}{3}$	
c)	$0° \le \theta \le 270°$	
d)		$\theta \in [0, \pi)$
e)		$\theta \in (0°, 450°)$
f)		$\theta \in (-2\pi, 4\pi]$

7. Solve for θ in the specified domain. Give solutions as exact values where possible. Otherwise, give approximate measures to the nearest thousandth.

   a) $2 \cos^2 \theta - 3 \cos \theta + 1 = 0, 0 \le \theta < 2\pi$

   b) $\tan^2 \theta - \tan \theta - 2 = 0, 0° \le \theta < 360°$

   c) $\sin^2 \theta - \sin \theta = 0, \theta \in [0, 2\pi)$

   d) $\sec^2 \theta - 2 \sec \theta - 3 = 0,$
   $\theta \in [-180°, 180°)$

8. Todd believes that 180° and 270° are solutions to the equation $5 \cos^2 \theta = -4 \cos \theta$. Show how you would check to determine whether Todd's solutions are correct.

Apply

9. Aslan and Shelley are finding the solution for $2 \sin^2 \theta = \sin \theta, 0 < \theta \le \pi$. Here is their work.

$2\sin^2 \theta = \sin \theta$

$\frac{2\sin^2 \theta}{\sin \theta} = \frac{\sin \theta}{\sin \theta}$ Step 1

$2\sin \theta = 1$ Step 2

$\sin \theta = \frac{1}{2}$ Step 3

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ Step 4

   a) Identify the error that Aslan and Shelley made and explain why their solution is incorrect.

   b) Show a correct method to determine the solution for $2 \sin^2 \theta = \sin \theta, 0 < \theta \le \pi$.

10. Explain why the equation $\sin \theta = 0$ has no solution in the interval $(\pi, 2\pi)$.

11. What is the solution for $\sin \theta = 2$? Show how you know. Does the interval matter?

12. Jaycee says that the trigonometric equation $\cos \theta = \frac{1}{2}$ has an infinite number of solutions. Do you agree? Explain.

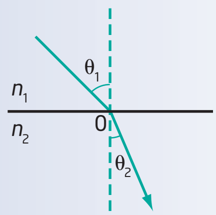
13. a) Helene is asked to solve the equation $3 \sin^2 \theta - 2 \sin \theta = 0, 0 \le \theta \le \pi$. She finds that $\theta = \pi$. Show how she could check whether this is a correct root for the equation.

    b) Find all the roots of the equation $3 \sin^2 \theta - 2 \sin \theta = 0, \theta \in [0, \pi]$.

14. Refer to the Did You Know? below. Use Snell's law of refraction to determine the angle of refraction of a ray of light passing from air into water if the angle of incidence is 35°. The refractive index is 1.000 29 for air and 1.33 for water.

Did You Know?

Willebrord Snell, a Dutch physicist, discovered that light is bent (refracted) as it passes from one medium into another. Snell's law is shown in the diagram.



$n_1 \sin \theta_1 = n_2 \sin \theta_2$,
where θ_1 is the angle of incidence,
θ_2 is the angle of refraction, and
n_1 and n_2 are the refractive indices of the mediums.

15. The average number of air conditioners sold in western Canada varies seasonally and depends on the month of the year. The formula $y = 5.9 + 2.4 \sin\left(\frac{\pi}{6}(t - 3)\right)$ gives the expected sales, y, in thousands, according to the month, t, where $t = 1$ represents January, $t = 2$ is February, and so on.

a) In what month are sales of 8300 air conditioners expected?

b) In what month are sales expected to be least?

c) Does this formula seem reasonable? Explain.

16. Nora is required to solve the following trigonometric equation.

$9 \sin^2 \theta + 12 \sin \theta + 4 = 0, \theta \in [0°, 360°)$

Nora did the work shown below. Examine her work carefully. Identify any errors. Rewrite the solution, making any changes necessary for it to be correct.

$$9 \sin^2 \theta + 12 \sin \theta + 4 = 0$$
$$(3 \sin \theta + 2)^2 = 0$$
$$3 \sin \theta + 2 = 0$$

Therefore, $\sin \theta = -\frac{2}{3}$

Use a calculator.

$$\sin^{-1}\left(-\frac{2}{3}\right) = -41.810\,314\,9$$

So, the reference angle is 41.8°, to the nearest tenth of a degree.

Sine is negative in quadrants II and III.

The solution in quadrant II is 180° − 41.8° = 138.2°.

The solution in quadrant III is 180° + 41.8° = 221.8°.

Therefore, θ = 138.2° and θ = 221.8°, to the nearest tenth of a degree.

17. Identify two different cases when a trigonometric equation would have no solution. Give an example to support each case.

18. Find the value of $\sec \theta$ if $\cot \theta = \frac{3}{4}$, $180° \le \theta \le 270°$.

Extend

19. A beach ball is riding the waves near Tofino, British Columbia. The ball goes up and down with the waves according to the formula $h = 1.4 \sin\left(\frac{\pi t}{3}\right)$, where h is the height, in metres, above sea level, and t is the time, in seconds.

a) In the first 10 s, when is the ball at sea level?

b) When does the ball reach its greatest height above sea level? Give the first time this occurs and then write an expression for every time the maximum occurs.

c) According to the formula, what is the most the ball goes below sea level?

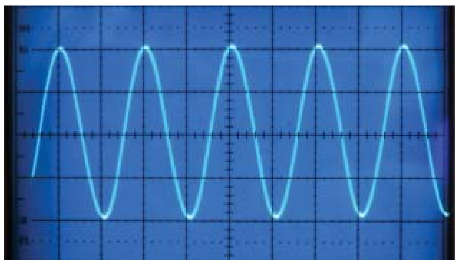
20. The current, I, in amperes, for an electric circuit is given by the formula $I = 4.3 \sin 120\pi t$, where t is time, in seconds.

a) The alternating current used in western Canada cycles 60 times per second. Demonstrate this using the given formula.

b) At what times is the current at its maximum value? How does your understanding of coterminal angles help in your solution?

c) At what times is the current at its minimum value?

d) What is the maximum current?

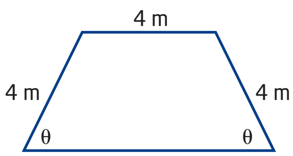


Oscilloscopes can measure wave functions of varying voltages.

21. Solve the trigonometric equation $\cos\left(x - \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}$, $-\pi < x < \pi$.

22. Consider the trigonometric equation $\sin^2 \theta + \sin \theta - 1 = 0$.
   a) Can you solve the equation by factoring?
   b) Use the quadratic formula to solve for $\sin \theta$.
   c) Determine all solutions for θ in the interval $0 < \theta \leq 2\pi$. Give answers to the nearest hundredth of a radian, if necessary.

23. Jaime plans to build a new deck behind her house. It is to be an isosceles trapezoid shape, as shown. She would like each outer edge of the deck to measure 4 m.



   a) Show that the area, A, of the deck is given by $A = 16 \sin \theta(1 + \cos \theta)$.
   b) Determine the exact value of θ in radians if the area of the deck is $12\sqrt{3}$ m^2.
   c) The angle in part b) gives the maximum area for the deck. How can you prove this? Compare your method with that of another student.

Create Connections

C1 Compare and contrast solving linear and quadratic equations with solving linear and quadratic trigonometric equations.

C2 A computer determines that a point on the unit circle has coordinates A(0.384 615 384 6, 0.923 076 923 1).
   a) How can you check whether a point is on the unit circle? Use your method to see if A is on the unit circle.
   b) If A is the point where the terminal arm of an angle θ intersects the unit circle, determine the values of $\cos \theta$, $\tan \theta$, and $\csc \theta$. Give your answers to three decimal places.
   c) Determine the measure of angle θ, to the nearest tenth of a degree. Does this approximate measure for θ seem reasonable for point A? Explain using a diagram.

C3 Use your knowledge of non-permissible values for rational expressions to answer the following.
   a) What is meant by the expression "non-permissible values"? Give an example.
   b) Use the fact that any point on the unit circle has coordinates $P(\theta) = (\cos \theta, \sin \theta)$ to identify a trigonometric relation that could have non-permissible values.
   c) For the trigonometric relation that you identified in part b), list all the values of θ in the interval $0 \leq \theta < 4\pi$ that are non-permissible.
   d) Create a general statement for all the non-permissible values of θ for your trigonometric relation over the real numbers.

C4 a) Determine all solutions for the equation $2 \sin^2 \theta = 1 - \sin \theta$ in the domain $0° \leq \theta < 360°$.
   b) Are your solutions exact or approximate? Why?
   c) Show how you can check one of your solutions to verify its correctness.

Chapter 4 Review

4.1 Angles and Angle Measure, pages 166–179

1. If each angle is in standard position, in which quadrant does it terminate?

a) $100°$

b) $500°$

c) 10

d) $\frac{29\pi}{6}$

2. Draw each angle in standard position. Convert each degree measure to radian measure and each radian measure to degree measure. Give answers as exact values.

a) $\frac{5\pi}{2}$

b) $240°$

c) $-405°$

d) -3.5

3. Convert each degree measure to radian measure and each radian measure to degree measure. Give answers as approximate values to the nearest hundredth, where necessary.

a) $20°$

b) $-185°$

c) -1.75

d) $\frac{5\pi}{12}$

4. Determine the measure of an angle coterminal with each angle in the domain $0° \leq \theta < 360°$ or $0 \leq \theta < 2\pi$. Draw a diagram showing the quadrant in which each angle terminates.

a) 6.75

b) $400°$

c) -3

d) $-105°$

5. Write an expression for all of the angles coterminal with each angle. Indicate what your variable represents.

a) $250°$

b) $\frac{5\pi}{2}$

c) $-300°$

d) 6

6. A jet engine motor cycle is tested at 80 000 rpm. What is this angular velocity in

a) radians per minute?

b) degrees per second?

4.2 The Unit Circle, pages 180–190

7. $P(\theta) = (x, y)$ is the point where the terminal arm of an angle θ intersects the unit circle. What are the coordinates for each point?

a) $P\left(\frac{5\pi}{6}\right)$

b) $P(-150°)$

c) $P\left(-\frac{11\pi}{2}\right)$

d) $P(45°)$

e) $P(120°)$

f) $P\left(\frac{11\pi}{3}\right)$

8. a) If the coordinates for $P\left(\frac{\pi}{3}\right)$ are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, explain how you can determine the coordinates for $P\left(\frac{2\pi}{3}\right)$, $P\left(\frac{4\pi}{3}\right)$, and $P\left(\frac{5\pi}{3}\right)$.

b) If the coordinates for $P(\theta)$ are $\left(-\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$, what are the coordinates for $P\left(\theta + \frac{\pi}{2}\right)$?

c) In which quadrant does $P\left(\frac{5\pi}{6} + \pi\right)$ lie? Explain how you know. If $P\left(\frac{5\pi}{6} + \pi\right)$ represents $P(\theta)$, what is the measure of θ and what are the coordinates of $P(\theta)$?

9. Identify all measures for θ in the interval $-2\pi \leq \theta < 2\pi$ such that $P(\theta)$ is the given point.

a) $(0, 1)$

b) $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

c) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

d) $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

10. Identify all measures for θ in the domain $-180° < \theta \leq 360°$ such that $P(\theta)$ is the given point.

a) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

b) $(-1, 0)$

c) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

d) $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

11. If $P(\theta) = \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\right)$, answer the following questions.

a) What is the measure of θ? Explain using a diagram.

b) In which quadrant does θ terminate?

c) What are the coordinates of $P(\theta + \pi)$?

d) What are the coordinates of $P\left(\theta + \frac{\pi}{2}\right)$?

e) What are the coordinates of $P\left(\theta - \frac{\pi}{2}\right)$?

4.3 Trigonometric Ratios, pages 191–205

12. If $\cos \theta = \frac{1}{3}$, $0° \leq \theta \leq 270°$, what is the value of each of the other trigonometric ratios of θ? When radicals occur, leave your answer in exact form.

13. Without using a calculator, determine the exact value of each trigonometric ratio.

a) $\sin\left(-\frac{3\pi}{2}\right)$

b) $\cos \frac{3\pi}{4}$

c) $\cot \frac{7\pi}{6}$

d) $\sec(-210°)$

e) $\tan 720°$

f) $\csc 300°$

14. Determine the approximate measure of all angles that satisfy the following. Give answers to the nearest hundredth of a unit. Draw a sketch to show the quadrant(s) involved.

a) $\sin \theta = 0.54$, $-2\pi < \theta \leq 2\pi$

b) $\tan \theta = 9.3$, $-180° \leq \theta < 360°$

c) $\cos \theta = -0.77$, $-\pi \leq \theta < \pi$

d) $\csc \theta = 9.5$, $-270° < \theta \leq 90°$

15. Determine each trigonometric ratio, to three decimal places.

a) $\sin 285°$

b) $\cot 130°$

c) $\cos 4.5$

d) $\sec 7.38$

16. The terminal arm of an angle θ in standard position passes through the point $A(-3, 4)$.

a) Draw the angle and use a protractor to determine its measure, to the nearest degree.

b) Show how to determine the exact value of $\cos \theta$.

c) What is the exact value of $\csc \theta + \tan \theta$?

d) From the value of $\cos \theta$, determine the measure of θ in degrees and in radians, to the nearest tenth.

4.4 Introduction to Trigonometric Equations, pages 206–214

17. Factor each trigonometric expression.

a) $\cos^2 \theta + \cos \theta$

b) $\sin^2 \theta - 3 \sin \theta - 4$

c) $\cot^2 \theta - 9$

d) $2 \tan^2 \theta - 9 \tan \theta + 10$

18. Explain why it is impossible to find each of the following values.

a) $\sin^{-1} 2$

b) $\tan 90°$

19. Without solving, determine the number of solutions for each trigonometric equation or statement in the specified domain.

a) $4 \cos \theta - 3 = 0, 0° < \theta \leq 360°$

b) $\sin \theta + 0.9 = 0, -\pi \leq \theta \leq \pi$

c) $0.5 \tan \theta - 1.5 = 0, -180° \leq \theta \leq 0°$

d) $\csc \theta$ is undefined, $\theta \in [-2\pi, 4\pi)$

20. Determine the exact roots for each trigonometric equation.

a) $\csc \theta = \sqrt{2}, \theta \in [0°, 360°]$

b) $2 \cos \theta + 1 = 0, 0 \leq \theta < 2\pi$

c) $3 \tan \theta - \sqrt{3} = 0, -180° \leq \theta < 360°$

d) $\cot \theta + 1 = 0, -\pi \leq \theta < \pi$

21. Solve for θ. Give solutions as exact values where possible. Otherwise, give approximate measures, to the nearest thousandth.

a) $\sin^2 \theta + \sin \theta - 2 = 0, 0 \leq \theta < 2\pi$

b) $\tan^2 \theta + 3 \tan \theta = 0, 0° < \theta \leq 360°$

c) $6 \cos^2 \theta + \cos \theta = 1, \theta \in (0°, 360°)$

d) $\sec^2 \theta - 4 = 0, \theta \in [-\pi, \pi]$

22. Determine a domain for which the equation $\sin \theta = \frac{\sqrt{3}}{2}$ would have the following solution.

a) $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

b) $\theta = -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}$

c) $\theta = -660°, -600°, -300°, -240°$

d) $\theta = -240°, 60°, 120°, 420°$

23. Determine each general solution using the angle measure specified.

a) $\sin x = -\frac{1}{2}$, in radians

b) $\sin x = \sin^2 x$, in degrees

c) $\sec x + 2 = 0$, in degrees

d) $(\tan x - 1)(\tan x - \sqrt{3}) = 0$, in radians

Chapter 4 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. If $\cos \theta = \frac{\sqrt{3}}{2}$, which could be the measure of θ?

A $\frac{2\pi}{3}$ **B** $\frac{5\pi}{6}$ **C** $\frac{5\pi}{3}$ **D** $\frac{11\pi}{6}$

2. Which exact measures of θ satisfy $\sin \theta = -\frac{\sqrt{3}}{2}$, $0° \leq \theta < 360°$?

A 60°, 120°

B −60°, −120°

C 240°, 300°

D −240°, −300°

3. If $\cot \theta = 1.4$, what is one approximate measure in radians for θ?

A 0.620

B 0.951

C 1.052

D 0.018

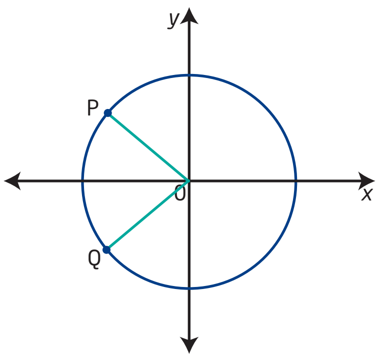
4. The coordinates of point P on the unit circle are $\left(-\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$. What are the coordinates of Q if Q is a 90° counterclockwise rotation from P?

A $\left(\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$

B $\left(-\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$

C $\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$

D $\left(-\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$



5. Determine the number of solutions for the trigonometric equation $\sin \theta (\sin \theta + 1) = 0$, $-180° < \theta < 360°$.

A 3

B 4

C 5

D 6

Short Answer

6. A vehicle has tires that are 75 cm in diameter. A point is marked on the edge of the tire.

a) Determine the measure of the angle through which the point turns every second if the vehicle is travelling at 110 km/h. Give your answer in degrees and in radians, to the nearest tenth.

b) What is the answer in radians if the diameter of the tire is 66 cm? Do you think that tire diameter affects tire life? Explain.

7. a) What is the equation for any circle with centre at the origin and radius 1 unit?

b) Determine the value(s) for the missing coordinate for all points on the unit circle satisfying the given conditions. Draw diagrams.

i) $\left(\frac{2\sqrt{3}}{5}, y\right)$

ii) $\left(x, \frac{\sqrt{7}}{4}\right)$, $x < 0$

c) Explain how to use the equation for the unit circle to find the value of cos θ if you know the *y*-coordinate of the point where the terminal arm of an angle θ in standard position intersects the unit circle.

8. Suppose that the cosine of an angle is negative and that you found one solution in quadrant III.
   a) Explain how to find the other solution between 0 and 2π.
   b) Describe how to write the general solution.

9. Solve the equation $2 \cos \theta + \sqrt{2} = 0$, where $\theta \in R$.

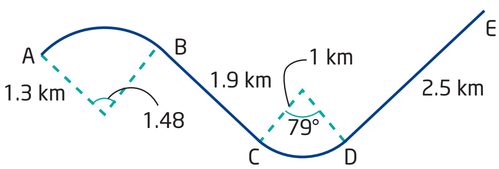
10. Explain the difference between an angle measuring 3° and one measuring 3 radians.

11. An angle in standard position measures $-500°$.
    a) In which quadrant does $-500°$ terminate?
    b) What is the measure of the reference angle?
    c) What is the approximate value, to one decimal place, of each trigonometric ratio for $-500°$?

12. Identify one positive and one negative angle measure that is coterminal with each angle. Then, write a general expression for all the coterminal angles in each case.
    a) $\frac{13\pi}{4}$
    b) $-575°$

Extended Response

13. The diagram shows a stretch of road from A to E. The curves are arcs of circles. Determine the length of the road from A to E. Give your answer to the nearest tenth of a kilometre.



14. Draw any $\triangle ABC$ with A at the origin, side AB along the positive x-axis, and C in quadrant I. Show that the area of your triangle can be expressed as $\frac{1}{2}bc \sin A$ or $\frac{1}{2}ac \sin B$.

15. Solve for θ. Give solutions as exact values where possible. Otherwise, give approximate measures to the nearest hundredth.
    a) $3 \tan^2 \theta - \tan \theta - 4 = 0$, $-\pi < \theta < 2\pi$
    b) $\sin^2 \theta + \sin \theta - 1 = 0$, $0 \le \theta < 2\pi$
    c) $\tan^2 \theta = 4 \tan \theta$, $\theta \in [0, 2\pi]$

16. Jack chooses a horse to ride on the West Edmonton Mall carousel. The horse is located 8 m from the centre of the carousel. If the carousel turns through an angle of 210° before stopping to let a crying child get off, how far did Jack travel? Give your answer as both an exact value and an approximate measure to the nearest hundredth of a metre.

CHAPTER 5

Trigonometric Functions and Graphs

You have seen different types of functions and how these functions can mathematically model the real world. Many sinusoidal and periodic patterns occur within nature. Movement on the surface of Earth, such as earthquakes, and stresses within Earth can cause rocks to fold into a sinusoidal pattern. Geologists and structural engineers study models of trigonometric functions to help them understand these formations. In this chapter, you will study trigonometric functions for which the function values repeat at regular intervals.

Key Terms

periodic function
period
sinusoidal curve
amplitude
vertical displacement
phase shift

Career Link

A geologist studies the composition, structure, and history of Earth's surface to determine the processes affecting the development of Earth. Geologists apply their knowledge of physics, chemistry, biology, and mathematics to explain these phenomena. Geological engineers apply geological knowledge to projects such as dam, tunnel, and building construction.

Web Link

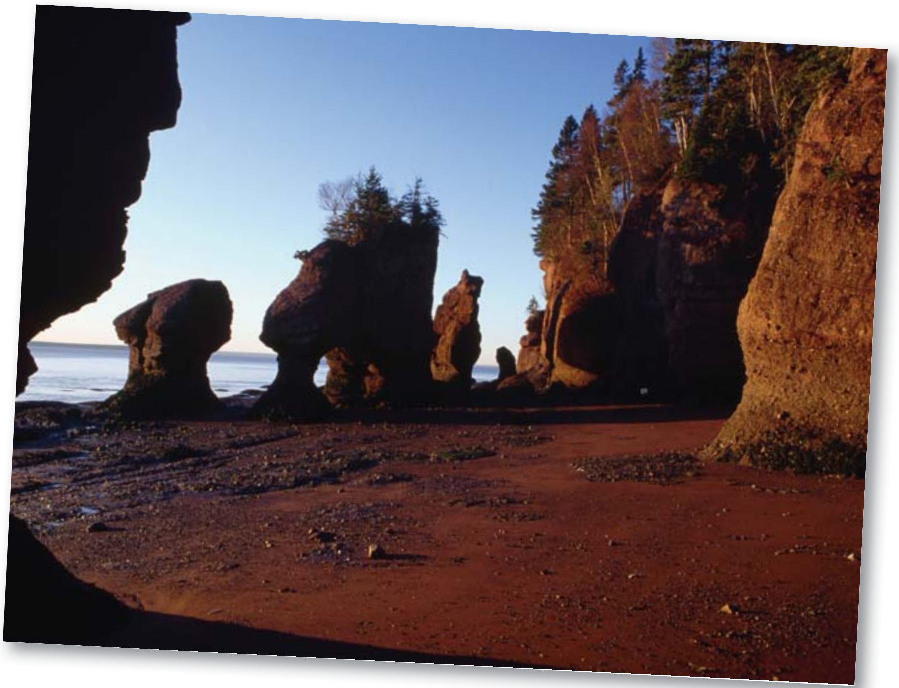
To learn more about a career as a geologist, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

5.1

Graphing Sine and Cosine Functions

Focus on...

- sketching the graphs of $y = \sin x$ and $y = \cos x$
- determining the characteristics of the graphs of $y = \sin x$ and $y = \cos x$
- demonstrating an understanding of the effects of vertical and horizontal stretches on the graphs of sinusoidal functions
- solving a problem by analysing the graph of a trigonometric function



The Hopewell Rocks on the Bay of Fundy coastline are sculpted by the cyclic tides.

Many natural phenomena are cyclic, such as the tides of the ocean, the orbit of Earth around the Sun, and the growth and decline in animal populations. What other examples of cyclic natural phenomena can you describe?

You can model these types of natural behaviour with periodic functions such as sine and cosine functions.

Did You Know?

The Bay of Fundy, between New Brunswick and Nova Scotia, has the highest tides in the world. The highest recorded tidal range is 17 m at Burntcoat Head, Nova Scotia.

Investigate the Sine and Cosine Functions

Materials

- grid paper
- ruler

1. a) Copy and complete the table. Use your knowledge of special angles to determine exact values for each trigonometric ratio. Then, determine the approximate values, to two decimal places. One row has been completed for you.

Angle, θ	$y = \sin \theta$	$y = \cos \theta$
0		
$\frac{\pi}{6}$	$\frac{1}{2} = 0.50$	$\frac{\sqrt{3}}{2} \approx 0.87$
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		

b) Extend the table to include multiples of the special angles in the other three quadrants.

2. a) Graph $y = \sin \theta$ on the interval $\theta \in [0, 2\pi]$

b) Summarize the following characteristics of the function $y = \sin \theta$.
- the maximum value and the minimum value
- the interval over which the pattern of the function repeats
- the zeros of the function in the interval $\theta \in [0, 2\pi]$
- the y-intercept
- the domain and range

3. Graph $y = \cos \theta$ on the interval $\theta \in [0, 2\pi]$ and create a summary similar to the one you developed in step 2b).

Reflect and Respond

4. a) Suppose that you extended the graph of $y = \sin \theta$ to the right of 2π. Predict the shape of the graph. Use a calculator to investigate a few points to the right of 2π. At what value of θ will the next cycle end?

b) Suppose that you extended the graph of $y = \sin \theta$ to the left of 0. Predict the shape of the graph. Use a calculator to investigate a few points to the left of 0. At what value of θ will the next cycle end?

5. Repeat step 4 for $y = \cos \theta$.

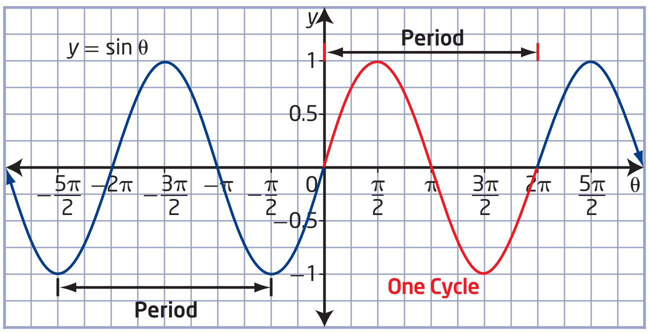
Did You Know?

The sine function is based upon one of the trigonometric ratios originally calculated by the astronomer Hipparchus of Nicaea in the second century B.C.E. He was trying to make sense of the movement of the stars and the moon in the night sky.

Link the Ideas

Sine and cosine functions are **periodic functions**. The values of these functions repeat over a specified **period**.

A sine graph is a graph of the function $y = \sin \theta$. You can also describe a sine graph as a **sinusoidal curve**.

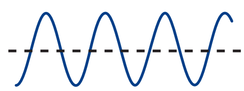


Trigonometric functions are sometimes called circular because they are based on the unit circle.

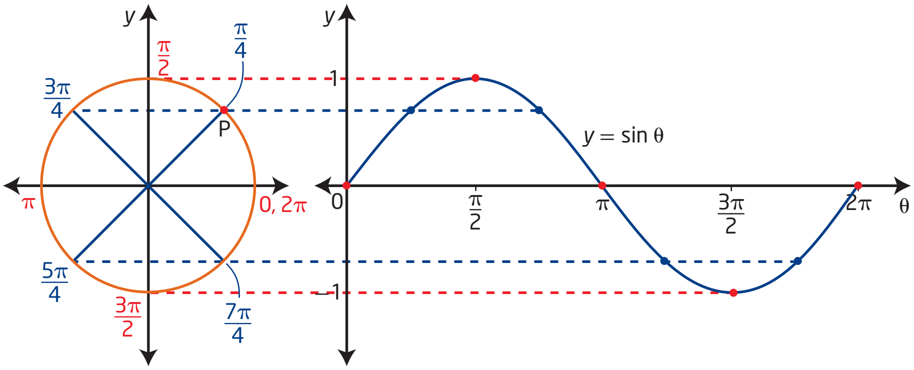
periodic function
- a function that repeats itself over regular intervals (cycles) of its domain

period
- the length of the interval of the domain over which a graph repeats itself
- the horizontal length of one cycle on a periodic graph

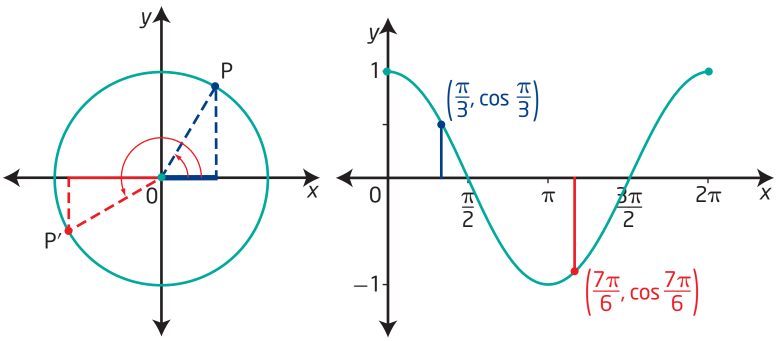
sinusoidal curve
- the name given to a curve that fluctuates back and forth like a sine graph
- a curve that oscillates repeatedly up and down from a centre line



The sine function, $y = \sin \theta$, relates the measure of angle θ in standard position to the y-coordinate of the point P where the terminal arm of the angle intersects the unit circle.



The cosine function, $y = \cos \theta$, relates the measure of angle θ in standard position to the x-coordinate of the point P where the terminal arm of the angle intersects the unit circle.



The coordinates of point P repeat after point P travels completely around the unit circle. The unit circle has a circumference of 2π. Therefore, the smallest distance before the cycle of values for the functions $y = \sin \theta$ or $y = \cos \theta$ begins to repeat is 2π. This distance is the period of $\sin \theta$ and $\cos \theta$.

Example 1

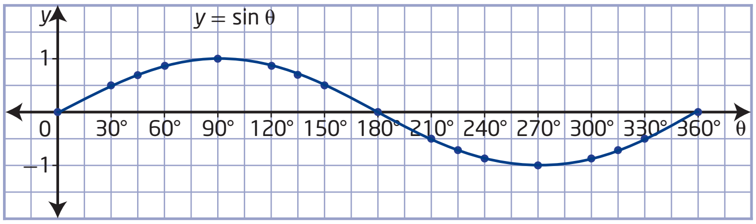
Graph a Periodic Function

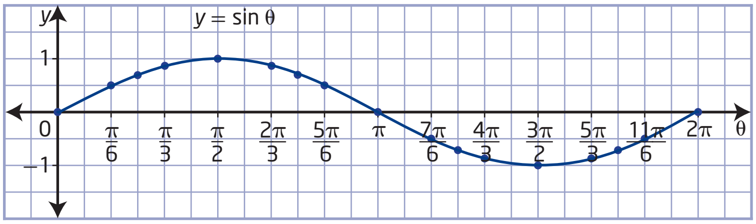
Sketch the graph of $y = \sin \theta$ for $0° \le \theta \le 360°$ or $0 \le \theta \le 2\pi$. Describe its characteristics.

Solution

To sketch the graph of the sine function for $0° \le \theta \le 360°$ or $0 \le \theta \le 2\pi$, select values of θ and determine the corresponding values of $\sin \theta$. Plot the points and join them with a smooth curve.

θ																		
θ	**Degrees**	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
θ	**Radians**	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin θ		0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0





From the graph of the sine function, you can make general observations about the characteristics of the sine curve:

- The curve is periodic.
- The curve is continuous.
- The domain is $\{\theta \mid \theta \in R\}$.
- The range is $\{y \mid -1 \le y \le 1, y \in R\}$.
- The maximum value is $+1$.
- The minimum value is -1.
- The **amplitude** of the curve is 1.
- The period is 360° or 2π.
- The y-intercept is 0.
- In degrees, the θ-intercepts are
  $\ldots, -540°, -360°, -180°, 0°, 180°, 360°, \ldots$, or $180°n$, where $n \in I$.
  The θ-intercepts, in radians, are
  $\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$, or $n\pi$, where $n \in I$.

Which points would you determine to be the key points for sketching a graph of the sine function?

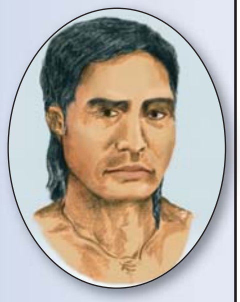
Look for a pattern in the values.

Your Turn

Sketch the graph of $y = \cos\theta$ for $0° \le \theta \le 360°$. Describe its characteristics.

Did You Know?

The Indo-Asian mathematician Aryabhata (476–550) made tables of half-chords that are now known as sine and cosine tables.



amplitude (of a sinusoidal function)

- the maximum vertical distance the graph of a sinusoidal function varies above and below the horizontal central axis of the curve

Example 2

Determine the Amplitude of a Sine Function

Any function of the form $y = af(x)$ is related to $y = f(x)$ by a vertical stretch of a factor $|a|$ about the x-axis, including the sine and cosine functions. If $a < 0$, the function is also reflected in the x-axis.

a) On the same set of axes, graph $y = 3 \sin x$, $y = 0.5 \sin x$, and $y = -2 \sin x$ for $0 \le x \le 2\pi$.

b) State the amplitude for each function.

c) Compare each graph to the graph of $y = \sin x$. Consider the period, amplitude, domain, and range.

Solution

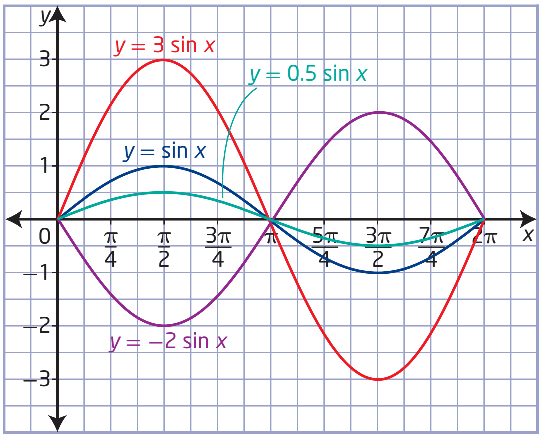
a) **Method 1: Graph Using Transformations**

Sketch the graph of $y = \sin x$.

For the graph of $y = 3 \sin x$, apply a vertical stretch by a factor of 3.

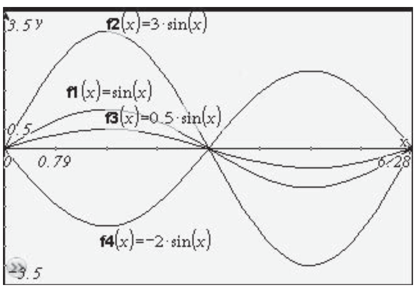
For the graph of $y = 0.5 \sin x$, apply a vertical stretch by a factor of 0.5.

For the graph of $y = -2 \sin x$, reflect in the x-axis and apply a vertical stretch by a factor of 2.



Method 2: Use a Graphing Calculator

Select radian mode.



Use the following window settings:
x: $\left[0, 2\pi, \frac{\pi}{4}\right]$
y: $[-3.5, 3.5, 0.5]$

b) Determine the amplitude of a sine function using the formula

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}.$$

How is the amplitude related to the range of the function?

The amplitude of $y = \sin x$ is $\frac{1 - (-1)}{2}$, or 1.

The amplitude of $y = 3 \sin x$ is $\frac{3 - (-3)}{2}$, or 3.

The amplitude of $y = 0.5 \sin x$ is $\frac{0.5 - (-0.5)}{2}$, or 0.5.

The amplitude of $y = -2 \sin x$ is $\frac{2 - (-2)}{2}$, or 2.

c)

Function	Period	Amplitude	Specified Domain	Range
$y = \sin x$	2π	1	$\{x \mid 0 \le x \le 2\pi, x \in \mathrm{R}\}$	$\{y \mid -1 \le y \le 1, y \in \mathrm{R}\}$
$y = 3 \sin x$	2π	3	$\{x \mid 0 \le x \le 2\pi, x \in \mathrm{R}\}$	$\{y \mid -3 \le y \le 3, y \in \mathrm{R}\}$
$y = 0.5 \sin x$	2π	0.5	$\{x \mid 0 \le x \le 2\pi, x \in \mathrm{R}\}$	$\{y \mid -0.5 \le y \le 0.5, y \in \mathrm{R}\}$
$y = -2 \sin x$	2π	2	$\{x \mid 0 \le x \le 2\pi, x \in \mathrm{R}\}$	$\{y \mid -2 \le y \le 2, y \in \mathrm{R}\}$

Changing the value of a affects the amplitude of a sinusoidal function. For the function $y = a \sin x$, the amplitude is $|a|$.

Your Turn

a) On the same set of axes, graph $y = 6 \cos x$ and $y = -4 \cos x$ for $0 \le x \le 2\pi$.

b) State the amplitude for each graph.

c) Compare your graphs to the graph of $y = \cos x$. Consider the period, amplitude, domain, and range.

d) What is the amplitude of the function $y = 1.5 \cos x$?

Period of $y = \sin bx$ or $y = \cos bx$

The graph of a function of the form $y = \sin bx$ or $y = \cos bx$ for $b \ne 0$ has a period different from 2π when $|b| \ne 1$. To show this, remember that $\sin bx$ or $\cos bx$ will take on all possible values as bx ranges from 0 to 2π. Therefore, to determine the period of either of these functions, solve the compound inequality as follows.

$0 \le x \le 2\pi$ Begin with the interval of one cycle of $y = \sin x$ or $y = \cos x$.

$0 \le |b|x \le 2\pi$ Replace x with $|b|x$ for the interval of one cycle of $y = \sin bx$ or $y = \cos bx$.

$0 \le x \le \frac{2\pi}{|b|}$ Divide by $|b|$.

Solving this inequality determines the length of a cycle for the sinusoidal curve, where the start of a cycle of $y = \sin bx$ is 0 and the end is $\frac{2\pi}{|b|}$.

Determine the period, or length of the cycle, by finding the distance from 0 to $\frac{2\pi}{|b|}$. Thus, the period for $y = \sin bx$ or $y = \cos bx$ is $\frac{2\pi}{|b|}$, in radians, or $\frac{360°}{|b|}$, in degrees.

Why do you use $|b|$ to determine the period?

Example 3

Determine the Period of a Sine Function

Any function of the form $y = f(bx)$ is related to $y = f(x)$ by a horizontal stretch by a factor of $\frac{1}{|b|}$ about the y-axis, including the sine and cosine functions. If $b < 0$, then the function is also reflected in the y-axis.

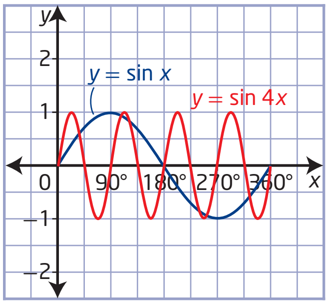
a) Sketch the graph of the function $y = \sin 4x$ for $0 \leq x \leq 360°$. State the period of the function and compare the graph to the graph of $y = \sin x$.

b) Sketch the graph of the function $y = \sin \frac{1}{2}x$ for $0 \leq x \leq 4\pi$. State the period of the function and compare the graph to the graph of $y = \sin x$.

Solution

a) Sketch the graph of $y = \sin x$.

For the graph of $y = \sin 4x$, apply a horizontal stretch by a factor of $\frac{1}{4}$.



To find the period of a function, start from any point on the graph (for example, the y-intercept) and determine the length of the interval until one cycle is complete.

From the graph of $y = \sin 4x$, the period is 90°.

You can also determine this using the formula Period $= \frac{360°}{|b|}$.

$$\text{Period} = \frac{360°}{|b|}$$

$$\text{Period} = \frac{360°}{|4|}$$

Substitute 4 for b.

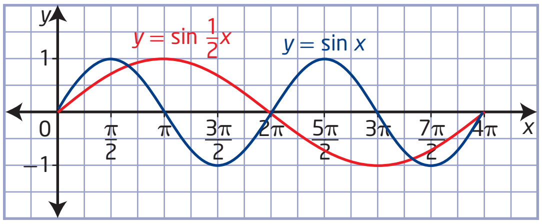
$$\text{Period} = \frac{360°}{4}$$

$$\text{Period} = 90°$$

Compared to the graph of $y = \sin x$, the graph of $y = \sin 4x$ has the same amplitude, domain, and range, but a different period.

b) Sketch the graph of $y = \sin x$.

For the graph of $y = \sin \frac{1}{2}x$, apply a horizontal stretch by a factor of 2.



From the graph, the period for $y = \sin \frac{1}{2}x$ is 4π.
Using the formula,

$$\text{Period} = \frac{2\pi}{|b|}$$

$$\text{Period} = \frac{2\pi}{\left|\frac{1}{2}\right|}$$ Substitute $\frac{1}{2}$ for b.

$$\text{Period} = \frac{2\pi}{\frac{1}{2}}$$

$$\text{Period} = 4\pi$$

Compared to the graph of $y = \sin x$, the graph of $y = \sin \frac{1}{2}x$ has the same amplitude, domain, and range, but a different period.

Changing the value of b affects the period of a sinusoidal function.

Your Turn

a) Sketch the graph of the function $y = \cos 3x$ for $0 \le x \le 360°$. State the period of the function and compare the graph to the graph of $y = \cos x$.

b) Sketch the graph of the function $y = \cos \frac{1}{3}x$ for $0 \le x \le 6\pi$. State the period of the function and compare the graph to the graph of $y = \cos x$.

c) What is the period of the graph of $y = \cos(-3x)$?

Example 4

Sketch the Graph of $y = a \cos bx$

a) Sketch the graph of $y = -3 \cos 2x$ for at least one cycle.

b) Determine
- the amplitude
- the period
- the maximum and minimum values
- the x-intercepts and the y-intercept
- the domain and range

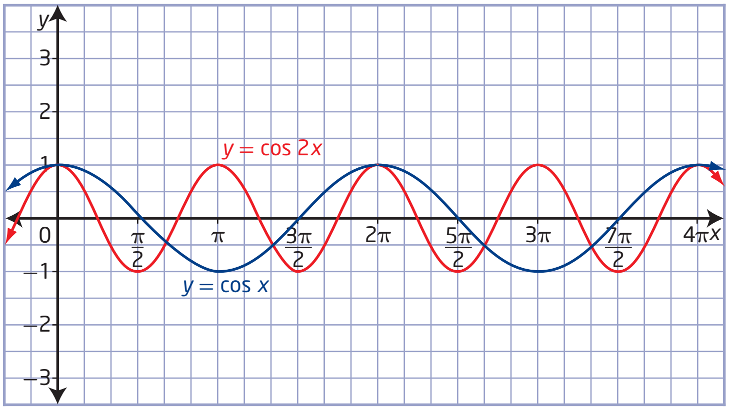
Solution

a) Method 1: Graph Using Transformations

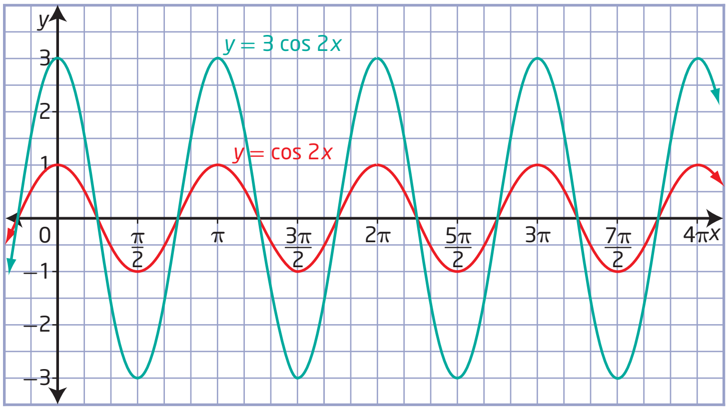
Compared to the graph of $y = \cos x$, the graph of $y = -3 \cos 2x$ is stretched horizontally by a factor of $\frac{1}{2}$ about the y-axis, stretched vertically by a factor of 3 about the x-axis, and reflected in the x-axis.

Begin with the graph of $y = \cos x$. Apply a horizontal stretch of $\frac{1}{2}$ about the y-axis.

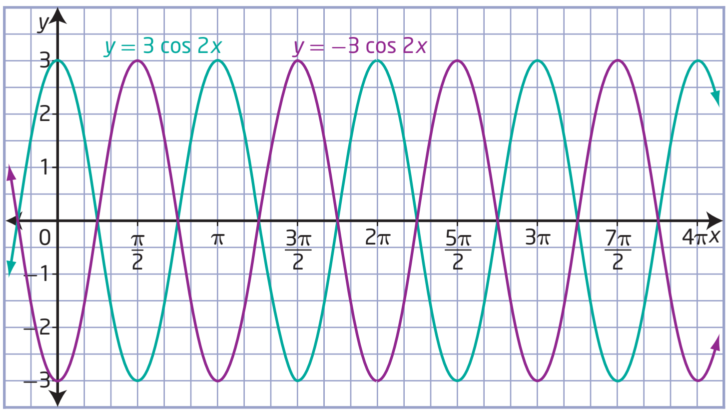
Why is the horizontal stretch by a factor of $\frac{1}{2}$?



Then, apply a vertical stretch by a factor of 3.



Finally, reflect the graph of $y = 3 \cos 2x$ in the x-axis.



Method 2: Graph Using Key Points

This method is based on the fact that one cycle of a cosine function $y = \cos bx$, from 0 to $\frac{2\pi}{|b|}$, includes two x-intercepts, two maximums, and a minimum. These five points divide the period into quarters.

Compare $y = -3 \cos 2x$ to $y = a \cos bx$.

Since $a = -3$, the amplitude is $|-3|$, or 3. Thus, the maximum value is 3 and the minimum value is -3.

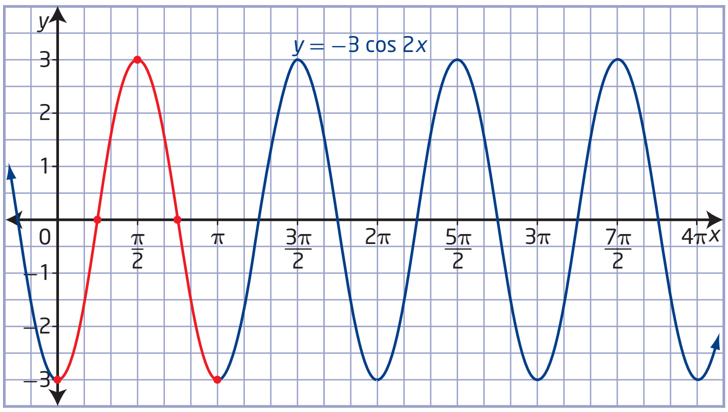
Since $b = 2$, the period is $\frac{2\pi}{|2|}$, or π. One cycle will start at $x = 0$ and end at $x = \pi$. Divide this cycle into four equal segments using the values $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π for x.

How do you know where the maximums or minimums will occur?

The key points are $(0, -3)$, $\left(\frac{\pi}{4}, 0\right)$, $\left(\frac{\pi}{2}, 3\right)$, $\left(\frac{3\pi}{4}, 0\right)$, and $(\pi, -3)$.

Why are there two minimums instead of two maximums?

Connect the points in a smooth curve and sketch the graph through one cycle. The graph of $y = -3 \cos 2x$ repeats every π units in either direction.



b) The amplitude of $y = -3 \cos 2x$ is 3.
The period is π.
The maximum value is 3.
The minimum value is -3
The y-intercept is -3.
The x-intercepts are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ or $\frac{\pi}{4} + \frac{\pi}{2}n$, $n \in \mathrm{I}$.
The domain of the function is $\{x \mid x \in \mathrm{R}\}$.
The range of the function is $\{y \mid -3 \leq y \leq 3, y \in \mathrm{R}\}$.

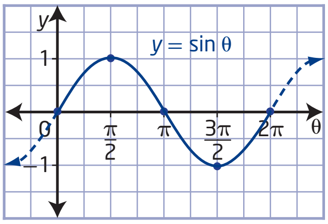
Your Turn

a) Graph $y = 3 \sin 4x$, showing at least two cycles.

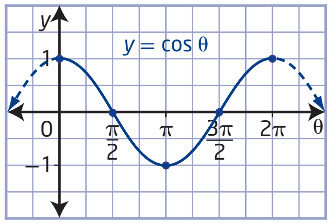
b) Determine
- the amplitude
- the period
- the maximum and minimum values
- the x-intercepts and the y-intercept
- the domain and range

Key Ideas

- To sketch the graphs of $y = \sin \theta$ and $y = \cos \theta$ for $0° \leq \theta \leq 360°$ or $0 \leq \theta \leq 2\pi$, determine the coordinates of the key points representing the θ-intercepts, maximum(s), and minimum(s).



The maximum value is $+1$.
The minimum value is -1.
The amplitude is 1.
The period is 2π.
The y-intercept is 0.
The θ-intercepts for the cycle shown are 0, π, and 2π.
The domain of $y = \sin \theta$ is $\{\theta \mid \theta \in \mathrm{R}\}$.
The range of $y = \sin \theta$ is $\{y \mid -1 \leq y \leq 1, y \in \mathrm{R}\}$.



How are the characteristics different for $y = \cos \theta$?

- Determine the amplitude and period of a sinusoidal function of the form $y = a \sin bx$ or $y = a \cos bx$ by inspecting graphs or directly from the sinusoidal function.
  - You can determine the amplitude using the formula
    $$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}.$$
    The amplitude is given by $|a|$.
    You can change the amplitude of a function by varying the value of a.
  - The period is the horizontal length of one cycle on the graph of a function. It is given by $\frac{2\pi}{|b|}$ or $\frac{360°}{|b|}$.
    You can change the period of a function by varying the value of b.

How can you determine the amplitude from the graph of the sine function? cosine function?

How can you identify the period on the graph of a sine function? cosine function?

Check Your Understanding

Practise

1. a) State the five key points for $y = \sin x$ that occur in one complete cycle from 0 to 2π.

   b) Use the key points to sketch the graph of $y = \sin x$ for $-2\pi \leq x \leq 2\pi$. Indicate the key points on your graph.

   c) What are the x-intercepts of the graph?

   d) What is the y-intercept of the graph?

   e) What is the maximum value of the graph? the minimum value?

2. a) State the five key points for $y = \cos x$ that occur in one complete cycle from 0 to 2π.

   b) Use the key points to sketch a graph of $y = \cos x$ for $-2\pi \leq x \leq 2\pi$. Indicate the key points on your graph.

   c) What are the x-intercepts of the graph?

   d) What is the y-intercept of the graph?

   e) What is the maximum value of the graph? the minimum value?

3. Copy and complete the table of properties for $y = \sin x$ and $y = \cos x$ for all real numbers.

Property	$y = \sin x$	$y = \cos x$
maximum		
minimum		
amplitude		
period		
domain		
range		
y-intercept		
x-intercepts		

4. State the amplitude of each periodic function. Sketch the graph of each function.

   a) $y = 2 \sin \theta$ b) $y = \frac{1}{2} \cos \theta$

   c) $y = -\frac{1}{3} \sin x$ d) $y = -6 \cos x$

5. State the period for each periodic function, in degrees and in radians. Sketch the graph of each function.

   a) $y = \sin 4\theta$ b) $y = \cos \frac{1}{3}\theta$

   c) $y = \sin \frac{2}{3}x$ d) $y = \cos 6x$

Apply

6. Match each function with its graph.

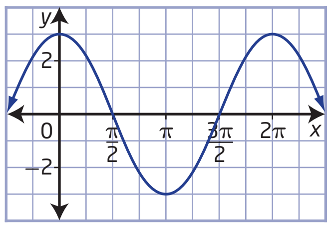
   a) $y = 3 \cos x$

   b) $y = \cos 3x$

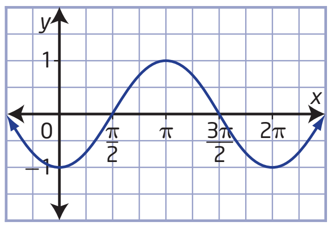
   c) $y = -\sin x$

   d) $y = -\cos x$

A



B



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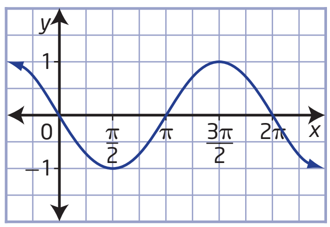

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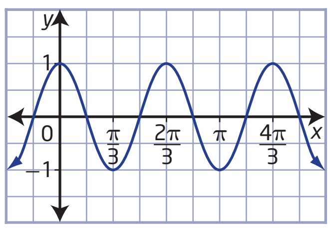


7. Determine the amplitude of each function. Then, use the language of transformations to describe how each graph is related to the graph of $y = \sin x$.

a) $y = 3 \sin x$ **b)** $y = -5 \sin x$

c) $y = 0.15 \sin x$ **d)** $y = -\frac{2}{3} \sin x$

8. Determine the period (in degrees) of each function. Then, use the language of transformations to describe how each graph is related to the graph of $y = \cos x$.

a) $y = \cos 2x$ **b)** $y = \cos (-3x)$

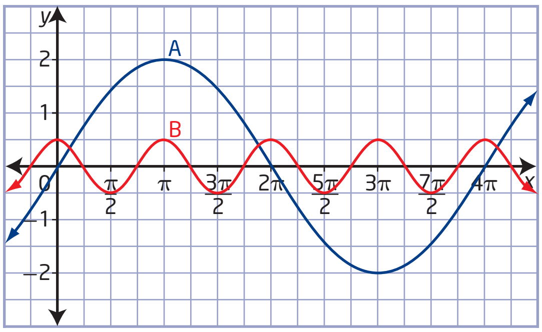
c) $y = \cos \frac{1}{4}x$ **d)** $y = \cos \frac{2}{3}x$

9. Without graphing, determine the amplitude and period of each function. State the period in degrees and in radians.

a) $y = 2 \sin x$ **b)** $y = -4 \cos 2x$

c) $y = \frac{5}{3} \sin \left(-\frac{2}{3}x\right)$ **d)** $y = 3 \cos \frac{1}{2}x$

10. a) Determine the period and the amplitude of each function in the graph.



b) Write an equation in the form $y = a \sin bx$ or $y = a \cos bx$ for each function.

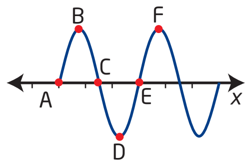
c) Explain your choice of either sine or cosine for each function.

11. Sketch the graph of each function over the interval [−360°, 360°]. For each function, clearly label the maximum and minimum values, the x-intercepts, the y-intercept, the period, and the range.

a) $y = 2 \cos x$ **b)** $y = -3 \sin x$

c) $y = \frac{1}{2} \sin x$ **d)** $y = -\frac{3}{4} \cos x$

12. The points indicated on the graph shown represent the x-intercepts and the maximum and minimum values.



a) Determine the coordinates of points B, C, D, and E if $y = 3 \sin 2x$ and A has coordinates (0, 0).

b) Determine the coordinates of points C, D, E, and F if $y = 2 \cos x$ and B has coordinates (0, 2).

c) Determine the coordinates of points B, C, D, and E if $y = \sin \frac{1}{2}x$ and A has coordinates $(-4\pi, 0)$.

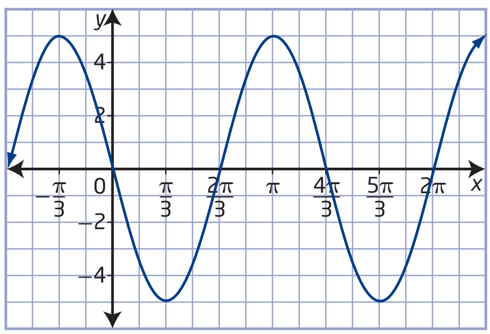
13. The second harmonic in sound is given by $f(x) = \sin 2x$, while the third harmonic is given by $f(x) = \sin 3x$. Sketch the curves and compare the graphs of the second and third harmonics for $-2\pi \le x \le 2\pi$.

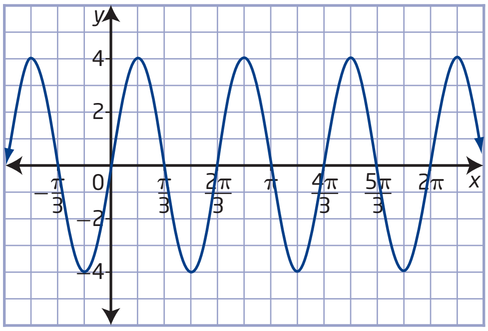
Did You Know?

A harmonic is a wave whose frequency is an integral multiple of the fundamental frequency. The fundamental frequency of a periodic wave is the inverse of the period length.

14. Sounds heard by the human ear are vibrations created by different air pressures. Musical sounds are regular or periodic vibrations. Pure tones will produce single sine waves on an oscilloscope. Determine the amplitude and period of each single sine wave shown.

a)

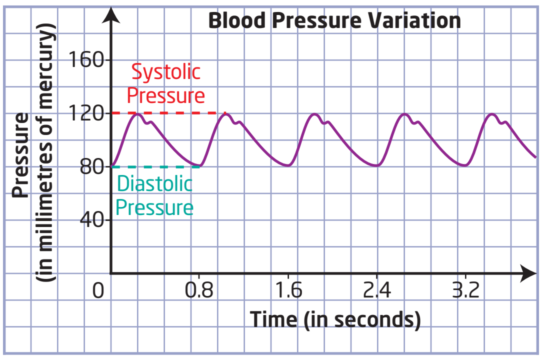


b) 

Did You Know?

Pure tone audiometry is a hearing test used to measure the hearing threshold levels of a patient. This test determines if there is hearing loss. Pure tone audiometry relies on a patient's response to pure tone stimuli.

15. Systolic and diastolic pressures mark the upper and lower limits in the changes in blood pressure that produce a pulse. The length of time between the peaks relates to the period of the pulse.



a) Determine the period and amplitude of the graph.

b) Determine the pulse rate (number of beats per minute) for this person.

16. **MINI LAB** Follow these steps to draw a sine curve.

Materials
- paper
- protractor
- compass
- ruler
- grid paper

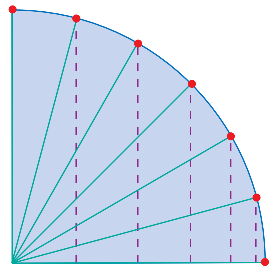
Step 1 Draw a large circle.

a) Mark the centre of the circle.

b) Use a protractor and mark every 15° from 0° to 180° along the circumference of the circle.

c) Draw a line radiating from the centre of the circle to each mark.

d) Draw a vertical line to complete a right triangle for each of the angles that you measured.



Step 2 Recall that the sine ratio is the length of the opposite side divided by the length of the hypotenuse. The hypotenuse of each triangle is the radius of the circle. Measure the length of the opposite side for each triangle and complete a table similar to the one shown.

Angle, x	Opposite	Hypotenuse	$\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$
0°			
15°			
30°			
45°			

Step 3 Draw a coordinate grid on a sheet of grid paper.

a) Label the x-axis from 0° to 360° in increments of 15°.

b) Label the y-axis from -1 to $+1$.

c) Create a scatter plot of points from your table. Join the dots with a smooth curve.

Step 4 Use one of the following methods to complete one cycle of the sine graph:

- complete the diagram from 180° to 360°
- extend the table by measuring the lengths of the sides of the triangle
- use the symmetry of the sine curve to complete the cycle

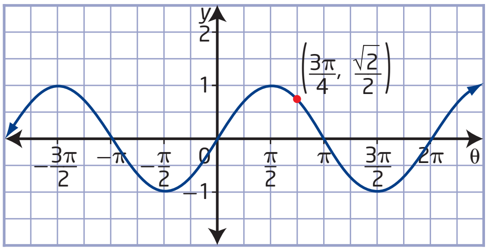
17. Sketch one cycle of a sinusoidal curve with the given amplitude and period and passing through the given point.

a) amplitude 2, period 180°, point (0, 0)

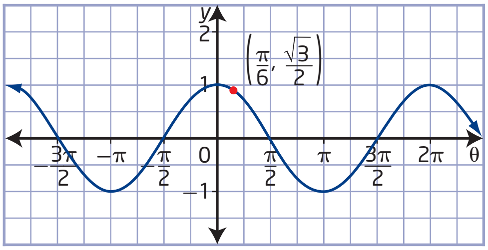
b) amplitude 1.5, period 540°, point (0, 0)

18. The graphs of $y = \sin\theta$ and $y = \cos\theta$ show the coordinates of one point. Determine the coordinates of four other points on the graph with the same y-coordinate as the point shown. Explain how you determined the θ-coordinates.

a)



b)



19. Graph $y = \sin\theta$ and $y = \cos\theta$ on the same set of axes for $-2\pi \le \theta \le 2\pi$.

a) How are the two graphs similar?

b) How are they different?

c) What transformation could you apply to make them the same graph?

Extend

20. If $y = f(x)$ has a period of 6, determine the period of $y = f\left(\frac{1}{2}x\right)$.

21. Determine the period, in radians, of each function using two different methods.

a) $y = -2\sin 3x$

b) $y = -\frac{2}{3}\cos\frac{\pi}{6}x$

22. If $\sin\theta = 0.3$, determine the value of $\sin\theta + \sin(\theta + 2\pi) + \sin(\theta + 4\pi)$.

23. Consider the function $y = \sqrt{\sin x}$.

a) Use the graph of $y = \sin x$ to sketch a prediction for the shape of the graph of $y = \sqrt{\sin x}$.

b) Use graphing technology or grid paper and a table of values to check your prediction. Resolve any differences.

c) How do you think the graph of $y = \sqrt{\sin x} + 1$ will differ from the graph of $y = \sqrt{\sin x}$?

d) Graph $y = \sqrt{\sin x} + 1$ and compare it to your prediction.

24. Is the function $f(x) = 5\cos x + 3\sin x$ sinusoidal? If it is sinusoidal, state the period of the function.

Did You Know?

In 1822, French mathematician Joseph Fourier discovered that any wave could be modelled as a combination of different types of sine waves. This model applies even to unusual waves such as square waves and highly irregular waves such as human speech. The discipline of reducing a complex wave to a combination of sine waves is called Fourier analysis and is fundamental to many of the sciences.

Create Connections

C1 MINI LAB Explore the relationship between the unit circle and the sine and cosine graphs with a graphing calculator.

Step 1 In the first list, enter the angle values from 0 to 2π by increments of $\frac{\pi}{12}$. In the second and third lists, calculate the cosine and sine of the angles in the first list, respectively.

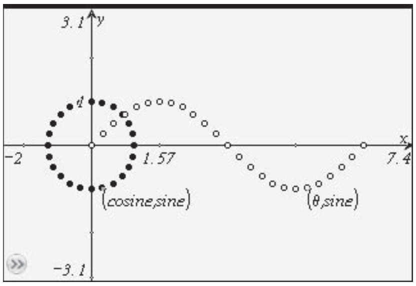
	A θ	B cosine	C sine	D	E
◆	=seq(x,x,0	=cos(θ)	=sin(θ)		
1	0.	1.	0.		
2	0.261799	0.965926	0.258819		
3	0.523599	0.866025	0.5		
4	0.785398	0.707107	0.707107		
5	1.0472	0.5	0.866025		
6	1.309	0.258819	0.965926		

A1 =0.

Step 2 Graph the second and third lists for the unit circle.

Step 3 Graph the first and third lists for the sine curve.

Step 4 Graph the first and second lists for the cosine curve.



Step 5 **a)** Use the trace feature on the graphing calculator and trace around the unit circle. What do you notice about the points that you trace? What do they represent?

b) Move the cursor to trace the sine or cosine curve. How do the points on the graph of the sine or cosine curve relate to the points on the unit circle? Explain.

C2 The value of $(\cos\theta)^2 + (\sin\theta)^2$ appears to be constant no matter the value of θ. What is the value of the constant? Why is the value constant? (Hint: Use the unit circle and the Pythagorean theorem in your explanation.)

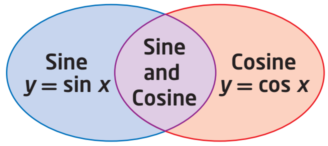
C3 The graph of $y = f(x)$ is sinusoidal with a period of 40° passing through the point (4, 0). Decide whether each of the following can be determined from this information, and justify your answer.

a) $f(0)$

b) $f(4)$

c) $f(84)$

C4 Identify the regions that each of the following characteristics fall into.



a) domain $\{x \mid x \in \mathrm{R}\}$

b) range $\{y \mid -1 \leq y \leq 1, y \in \mathrm{R}\}$

c) period is 2π

d) amplitude is 1

e) x-intercepts are $n(180°)$, $n \in \mathrm{I}$

f) x-intercepts are $90° + n(180°)$, $n \in \mathrm{I}$

g) y-intercept is 1

h) y-intercept is 0

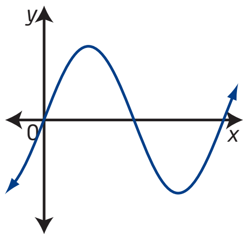
i) passes through point (0, 1)

j) passes through point (0, 0)

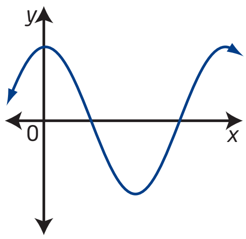
k) a maximum value occurs at (360°, 1)

l) a maximum value occurs at (90°, 1)

m)



n)



C5 **a)** Sketch the graph of $y = |\cos x|$ for $-2\pi \leq x \leq 2\pi$. How does the graph compare to the graph of $y = \cos x$?

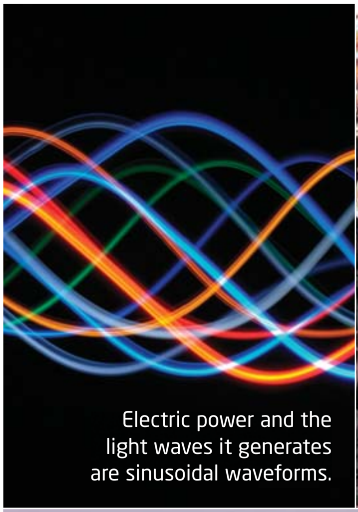
b) Sketch the graph of $y = |\sin x|$ for $-2\pi \leq x \leq 2\pi$. How does the graph compare to the graph of $y = \sin x$?

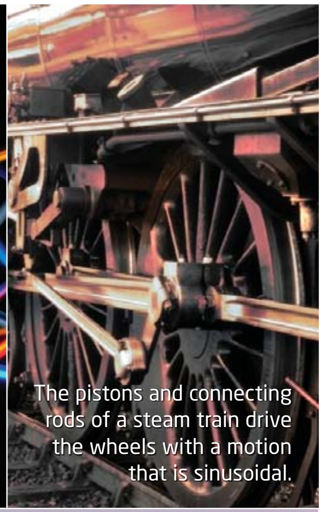
5.2

Transformations of Sinusoidal Functions

Focus on...

- graphing and transforming sinusoidal functions
- identifying the domain, range, phase shift, period, amplitude, and vertical displacement of sinusoidal functions
- developing equations of sinusoidal functions, expressed in radian and degree measure, from graphs and descriptions
- solving problems graphically that can be modelled using sinusoidal functions
- recognizing that more than one equation can be used to represent the graph of a sinusoidal function


Electric power and the light waves it generates are sinusoidal waveforms.


The pistons and connecting rods of a steam train drive the wheels with a motion that is sinusoidal.


Ocean waves created by the winds may be modelled by sinusoidal curves.

The motion of a body attached to a suspended spring, the motion of the plucked string of a musical instrument, and the pendulum of a clock produce oscillatory motion that you can model with sinusoidal functions. To use the functions $y = \sin x$ and $y = \cos x$ in applied situations, such as these and the ones in the images shown, you need to be able to transform the functions.

Investigate Transformations of Sinusoidal Functions

Materials

- grid paper
- graphing technology

A: Graph $y = \sin \theta + d$ or $y = \cos \theta + d$

1. On the same set of axes, sketch the graphs of the following functions for $0° \le \theta \le 360°$.

$y = \sin \theta$
$y = \sin \theta + 1$
$y = \sin \theta - 2$

2. Using the language of transformations, compare the graphs of $y = \sin \theta + 1$ and $y = \sin \theta - 2$ to the graph of $y = \sin \theta$.

3. Predict what the graphs of $y = \sin \theta + 3$ and $y = \sin \theta - 4$ will look like. Justify your predictions.

Reflect and Respond

4. a) What effect does the parameter d in the function $y = \sin \theta + d$ have on the graph of $y = \sin \theta$ when $d > 0$?

b) What effect does the parameter d in the function $y = \sin \theta + d$ have on the graph of $y = \sin \theta$ when $d < 0$?

5. a) Predict the effect varying the parameter d in the function $y = \cos \theta + d$ has on the graph of $y = \cos \theta$.

b) Use a graph to verify your prediction.

B: Graph $y = \cos(\theta - c)$ or $y = \sin(\theta - c)$ Using Technology

6. On the same set of axes, sketch the graphs of the following functions for $-\pi \leq \theta \leq 2\pi$.

$y = \cos \theta$

$y = \cos\left(\theta + \frac{\pi}{2}\right)$

$y = \cos(\theta - \pi)$

7. Using the language of transformations, compare the graphs of $y = \cos\left(\theta + \frac{\pi}{2}\right)$ and $y = \cos(\theta - \pi)$ to the graph of $y = \cos \theta$.

8. Predict what the graphs of $y = \cos\left(\theta - \frac{\pi}{2}\right)$ and $y = \cos\left(\theta + \frac{3\pi}{2}\right)$ will look like. Justify your predictions.

Reflect and Respond

9. a) What effect does the parameter c in the function $y = \cos(\theta - c)$ have on the graph of $y = \cos \theta$ when $c > 0$?

b) What effect does the parameter c in the function $y = \cos(\theta - c)$ have on the graph of $y = \cos \theta$ when $c < 0$?

10. a) Predict the effect varying the parameter c in the function $y = \sin(\theta - c)$ has on the graph of $y = \sin \theta$.

b) Use a graph to verify your prediction.

Link the Ideas

You can translate graphs of functions up or down or left or right and stretch them vertically and/or horizontally. The rules that you have applied to the transformations of functions also apply to transformations of sinusoidal curves.

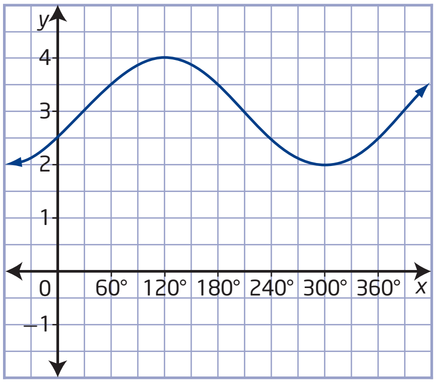
Example 1

Graph $y = \sin (x - c) + d$

a) Sketch the graph of the function $y = \sin (x - 30°) + 3$.
b) What are the domain and range of the function?
c) Use the language of transformations to compare your graph to the graph of $y = \sin x$.

Solution

a)



b) Domain: $\{x \mid x \in \mathrm{R}\}$
Range: $\{y \mid 2 \le y \le 4, y \in \mathrm{R}\}$

c) The graph has been translated 3 units up. This is the **vertical displacement**. The graph has also been translated 30° to the right. This is called the **phase shift**.

vertical displacement

- the vertical translation of the graph of a periodic function

phase shift

- the horizontal translation of the graph of a periodic function

Your Turn

a) Sketch the graph of the function $y = \cos (x + 45°) - 2$.
b) What are the domain and range of the function?
c) Use the language of transformations to compare your graph to the graph of $y = \cos x$.

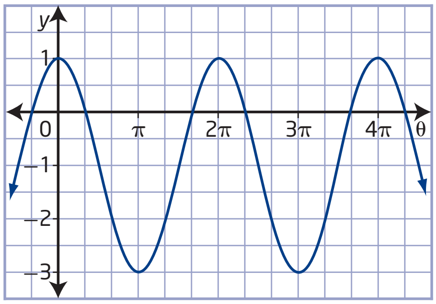
Example 2

Graph $y = a \cos (\theta - c) + d$

a) Sketch the graph of the function $y = -2 \cos (\theta + \pi) - 1$ over two cycles.
b) Use the language of transformations to compare your graph to the graph of $y = \cos \theta$. Indicate which parameter is related to each transformation.

Solution

a)



b) Since a is -2, the graph has been reflected about the θ-axis and then stretched vertically by a factor of two. The d-value is -1, so the graph is translated 1 unit down. The sinusoidal axis is defined as $y = -1$. Finally, the c-value is $-\pi$. Therefore, the graph is translated π units to the left.

Your Turn

a) Sketch the graph of the function $y = 2 \sin \left(\theta - \frac{\pi}{2}\right) + 2$ over two cycles.

b) Compare your graph to the graph of $y = \sin \theta$.

Did You Know?

In this chapter, the parameters for horizontal and vertical translations are represented by c and d, respectively.

Example 3

Graph $y = a \sin b(x - c) + d$

Sketch the graph of the function $y = 3 \sin \left(2x - \frac{2\pi}{3}\right) + 2$ over two cycles. What are the vertical displacement, amplitude, period, phase shift, domain, and range for the function?

Solution

First, rewrite the function in the standard form $y = a \sin b(x - c) + d$.

$y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$

Method 1: Graph Using Transformations

Step 1: Sketch the graph of $y = \sin x$ for one cycle. Apply the horizontal and vertical stretches to obtain the graph of $y = 3 \sin 2x$.

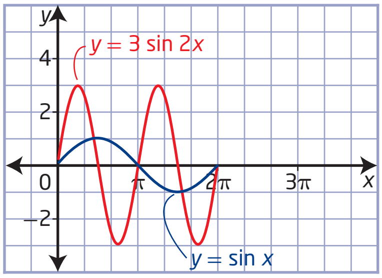
Compared to the graph of $y = \sin x$, the graph of $y = 3 \sin 2x$ is a horizontal stretch by a factor of $\frac{1}{2}$ and a vertical stretch by a factor of 3.

For the function $y = 3 \sin 2x$, $b = 2$.

$$\begin{aligned} \text{Period} &= \frac{2\pi}{|b|} \\ &= \frac{2\pi}{2} \\ &= \pi \end{aligned}$$

So, the period is π.

For the function $y = 3 \sin 2x$, $|a| = 3$.
So, the amplitude is 3.

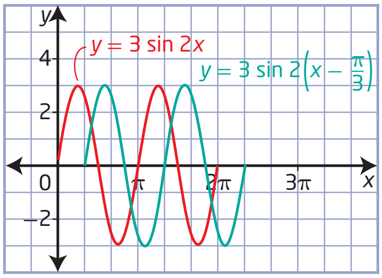


Step 2: Apply the horizontal translation to obtain the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right)$.

The phase shift is determined by the value of parameter c for a function in the standard form $y = a \sin b(x - c) + d$.

Compared to the graph of $y = 3 \sin 2x$, the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right)$ is translated horizontally $\frac{\pi}{3}$ units to the right.

The phase shift is $\frac{\pi}{3}$ units to the right.

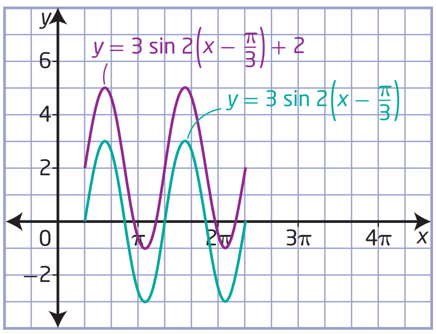


Step 3: Apply the vertical translation to obtain the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$.

The vertical displacement is determined by the value of parameter d for a function in the standard form $y = a \sin b(x - c) + d$.

Compared to the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right)$, the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$ is translated up 2 units.

The vertical displacement is 2 units up.



Would it matter if the order of the transformations were changed? Try a different order for the transformations.

Compared to the graph of $y = \sin x$, the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$ is

- horizontally stretched by a factor of $\frac{1}{2}$
- vertically stretched by a factor of 3
- horizontally translated $\frac{\pi}{3}$ units to the right
- vertically translated 2 units up

The vertical displacement is 2 units up.
The amplitude is 3.
The phase shift is $\frac{\pi}{3}$ units to the right.
The domain is $\{x \mid x \in \mathrm{R}\}$.
The range is $\{y \mid -1 \leq y \leq 5, y \in \mathrm{R}\}$.

Method 2: Graph Using Key Points
You can identify five key points to graph one cycle of the sine function. The first, third, and fifth points indicate the start, the middle, and the end of the cycle. The second and fourth points indicate the maximum and minimum points.

Comparing $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$ to $y = a \sin b(x - c) + d$ gives $a = 3$, $b = 2$, $c = \frac{\pi}{3}$, and $d = 2$.

The amplitude is $|a|$, or 3.

The period is $\frac{2\pi}{|b|}$, or π.

The vertical displacement is d, or 2. Therefore, the equation of the sinusoidal axis or mid-line is $y = 2$.

You can use the amplitude and vertical displacement to determine the maximum and minimum values.

The maximum value is
$$\begin{aligned} d + |a| &= 2 + 3 \\ &= 5 \end{aligned}$$

The minimum value is
$$\begin{aligned} d - |a| &= 2 - 3 \\ &= -1 \end{aligned}$$

Determine the values of x for the start and end of one cycle from the function $y = a \sin b(x - c) + d$ by solving the compound inequality $0 \leq b(x - c) \leq 2\pi$.

$$0 \leq 2\left(x - \frac{\pi}{3}\right) \leq 2\pi$$

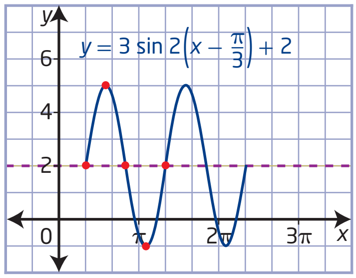
How does this inequality relate to the period of the function?

$$0 \leq x - \frac{\pi}{3} \leq \pi$$
$$\frac{\pi}{3} \leq x \leq \frac{4\pi}{3}$$

Divide the interval $\frac{\pi}{3} \leq x \leq \frac{4\pi}{3}$ into four equal segments. By doing this, you can locate five key values of x along the sinusoidal axis.

$\frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}, \frac{13\pi}{12}, \frac{4\pi}{3}$

Use the above information to sketch one cycle of the graph, and then a second cycle.



Note the five key points and how you can use them to sketch one cycle of the graph of the function.

For the graph of the function $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$,

- the vertical displacement is 2 units up
- the amplitude is 3
- the phase shift is $\frac{\pi}{3}$ units to the right
- the domain is $\{x \mid x \in R\}$
- the range is $\{y \mid -1 \leq y \leq 5, y \in R\}$

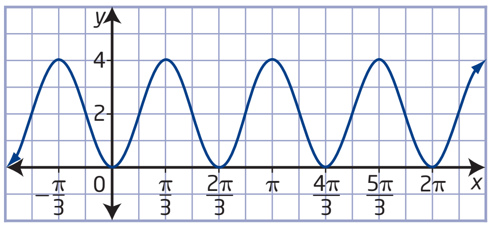
Your Turn

Sketch the graph of the function $y = 2 \cos 4(x + \pi) - 1$ over two cycles. What are the vertical displacement, amplitude, period, phase shift, domain, and range for the function?

Example 4

Determine an Equation From a Graph

The graph shows the function $y = f(x)$.



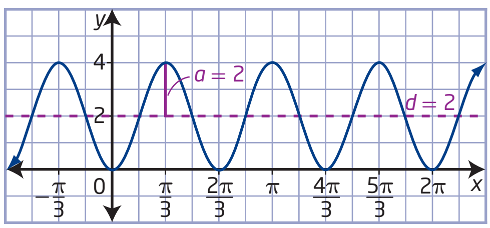
a) Write the equation of the function in the form $y = a \sin b(x - c) + d$, $a > 0$.

b) Write the equation of the function in the form $y = a \cos b(x - c) + d$, $a > 0$.

c) Use technology to verify your solutions.

Solution

a) Determine the values of the parameters a, b, c, and d.



Locate the sinusoidal axis or mid-line. Its position determines the value of d. Thus, $d = 2$.

Use the sinusoidal axis from the graph or use the formula to determine the amplitude.

How can you use the maximum and minimum values of the graph to find the value of d?

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$

$$a = \frac{4 - 0}{2}$$

$$a = 2$$

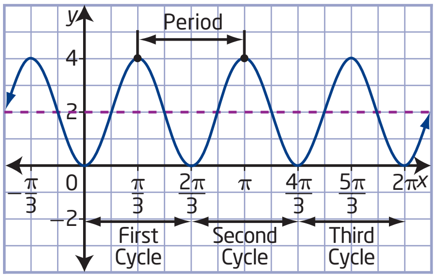
The amplitude is 2.

Determine the period and the value of b.

Method 1: Count the Number of Cycles in 2π

Determine the number of cycles in a distance of 2π.

In this function, there are three cycles. Therefore, the value of b is 3 and the period is $\frac{2\pi}{3}$.



Method 2: Determine the Period First

Locate the start and end of one cycle of the sine curve.

Recall that one cycle of $y = \sin x$ starts at (0, 0). How is that point transformed? How could this information help you determine the start for one cycle of this sine curve?

The start of the first cycle of the sine curve that is closest to the y-axis is at $x = \frac{\pi}{6}$ and the end is at $x = \frac{5\pi}{6}$.

The period is $\frac{5\pi}{6} - \frac{\pi}{6}$, or $\frac{2\pi}{3}$.

Solve the equation for b.

$$\text{Period} = \frac{2\pi}{|b|}$$

$$\frac{2\pi}{3} = \frac{2\pi}{|b|}$$

$$b = 3$$

Choose b to be positive.

Determine the phase shift, c.

Locate the start of the first cycle of the sine curve to the right of the y-axis. Thus, $c = \frac{\pi}{6}$.

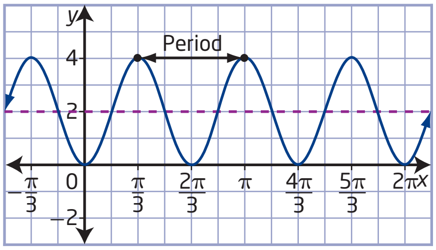
Substitute the values of the parameters $a = 2$, $b = 3$, $c = \frac{\pi}{6}$, and $d = 2$ into the equation $y = a \sin b(x - c) + d$.

The equation of the function in the form $y = a \sin b(x - c) + d$ is $y = 2 \sin 3\left(x - \frac{\pi}{6}\right) + 2$.

b) To write an equation in the form $y = a \cos b(x - c) + d$, determine the values of the parameters a, b, c, and d using steps similar to what you did for the sine function in part a).

$a = 2$
$b = 3$
$c = \frac{\pi}{3}$
$d = 2$

Why is $c = \frac{\pi}{3}$? Are there other possible values for c?

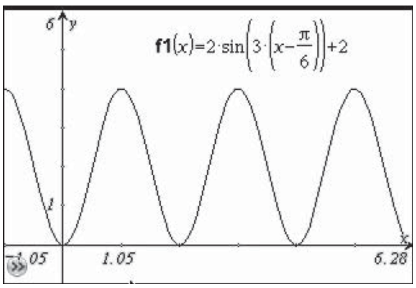


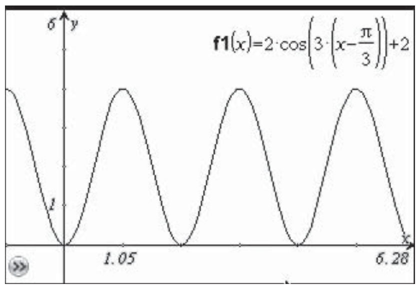
The equation of the function in the form $y = a \cos b(x - c) + d$ is $y = 2 \cos 3\left(x - \frac{\pi}{3}\right) + 2$.

How do the two equations compare?

Could other equations define the function $y = f(x)$?

c) Enter the functions on a graphing calculator. Compare the graphs to the original and to each other.

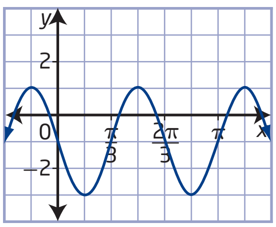




The graphs confirm that the equations for the function are correct.

Your Turn

The graph shows the function $y = f(x)$.



a) Write the equation of the function in the form $y = a \sin b(x - c) + d$, $a > 0$.
b) Write the equation of the function in the form $y = a \cos b(x - c) + d$, $a > 0$.
c) Use technology to verify your solutions.

Example 5

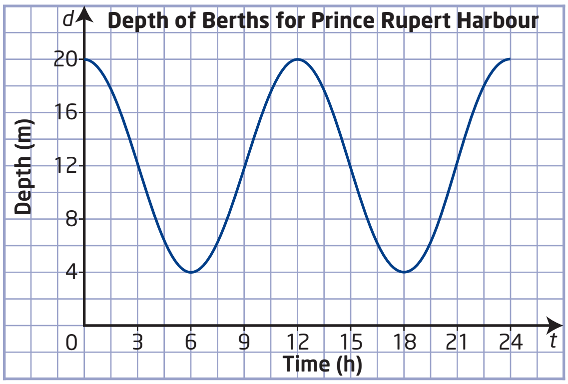
Interpret Graphs of Sinusoidal Functions

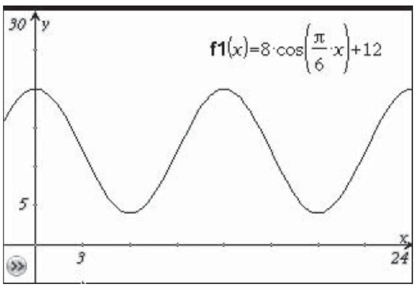
Prince Rupert, British Columbia, has the deepest natural harbour in North America. The depth, d, in metres, of the berths for the ships can be approximated by the equation $d(t) = 8 \cos \frac{\pi}{6}t + 12$, where t is the time, in hours, after the first high tide.

a) Graph the function for two cycles.
b) What is the period of the tide?
c) An ocean liner requires a minimum of 13 m of water to dock safely. From the graph, determine the number of hours per cycle the ocean liner can safely dock.
d) If the minimum depth of the berth occurs at 6 h, determine the depth of the water. At what other times is the water level at a minimum? Explain your solution.

Solution

a)





Why should you set the calculator to radian mode when graphing sinusoidal functions that represent real-world situations?

b) Use $b = \frac{\pi}{6}$ to determine the period.

$\text{Period} = \frac{2\pi}{|b|}$

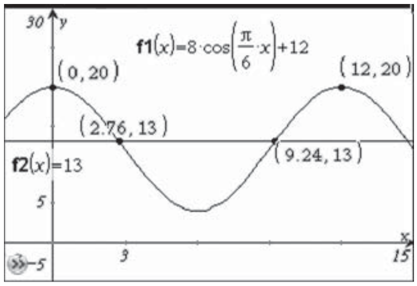
$\text{Period} = \frac{2\pi}{\left|\frac{\pi}{6}\right|}$

$\text{Period} = 12$

The period for the tides is 12 h.

What does the period of 12 h represent?

c) To determine the number of hours an ocean liner can dock safely, draw the line $y = 13$ to represent the minimum depth of the berth. Determine the points of intersection of the graphs of $y = 13$ and $d(t) = 8 \cos \frac{\pi}{6}t + 12$.



More precise answers can be obtained using technology.

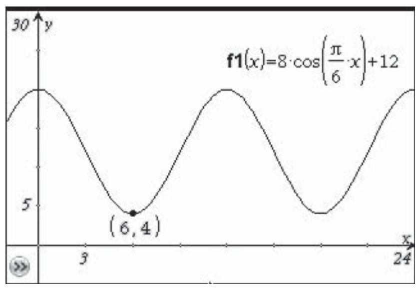
The points of intersection for the first cycle are approximately (2.76, 13) and (9.26, 13).

The depth is greater than 13 m from 0 h to approximately 2.76 h and from approximately 9.24 h to 12 h. The total time when the depth is greater than 13 m is 2.76 + 2.76, or 5.52 h, or about 5 h 30 min per cycle.

d) To determine the berth depth at 6 h, substitute the value of $t = 6$ into the equation.

$$d(t) = 8 \cos \frac{\pi}{6}t + 12$$
$$d(6) = 8 \cos \frac{\pi}{6}(6) + 12$$
$$d(6) = 8 \cos \pi + 12$$
$$d(6) = 8(-1) + 12$$
$$d(6) = 4$$

You can use the graph to verify the solution.



The berth depth at 6 h is 4 m. Add 12 h (the period) to 6 h to determine the next time the berth depth is 4 m. Therefore, the berth depth of 4 m occurs again at 18 h.

Your Turn

The depth, d, in metres, of the water in the harbour at New Westminster, British Columbia, is approximated by the equation $d(t) = 0.6 \cos \frac{2\pi}{13}t + 3.7$, where t is the time, in hours, after the first high tide.

a) Graph the function for two cycles starting at $t = 0$.

b) What is the period of the tide?

c) If a boat requires a minimum of 3.5 m of water to launch safely, for how many hours per cycle can the boat safely launch?

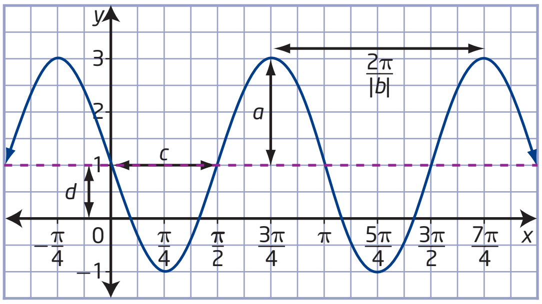
d) What is the depth of the water at 7 h? At what other times is the water level at this depth? Explain your solution.

Key Ideas

- You can determine the amplitude, period, phase shift, and vertical displacement of sinusoidal functions when the equation of the function is given in the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$.

  For: $y = a \sin b(x - c) + d$
  $y = a \cos b(x - c) + d$

  How does changing each parameter affect the graph of a function?

  Vertical stretch by a factor of $|a|$
  - changes the amplitude to $|a|$
  - reflected in the x-axis if $a < 0$

  Horizontal stretch by a factor of $\frac{1}{|b|}$
  - changes the period to $\frac{360°}{|b|}$ (in degrees) or $\frac{2\pi}{|b|}$ (in radians)
  - reflected in the y-axis if $b < 0$

  Horizontal phase shift represented by c
  - to right if $c > 0$
  - to left if $c < 0$

  Vertical displacement represented by d
  - up if $d > 0$
  - down if $d < 0$

  $$d = \frac{\text{maximum value} + \text{minimum value}}{2}$$

- You can determine the equation of a sinusoidal function given its properties or its graph.

Check Your Understanding

Practise

1. Determine the phase shift and the vertical displacement with respect to $y = \sin x$ for each function. Sketch a graph of each function.
   a) $y = \sin (x - 50°) + 3$
   b) $y = \sin (x + \pi)$
   c) $y = \sin \left(x + \frac{2\pi}{3}\right) + 5$
   d) $y = 2 \sin (x + 50°) - 10$
   e) $y = -3 \sin (6x + 30°) - 3$
   f) $y = 3 \sin \frac{1}{2}\left(x - \frac{\pi}{4}\right) - 10$

2. Determine the phase shift and the vertical displacement with respect to $y = \cos x$ for each function. Sketch a graph of each function.
   a) $y = \cos (x - 30°) + 12$
   b) $y = \cos \left(x - \frac{\pi}{3}\right)$
   c) $y = \cos \left(x + \frac{5\pi}{6}\right) + 16$
   d) $y = 4 \cos (x + 15°) + 3$
   e) $y = 4 \cos (x - \pi) + 4$
   f) $y = 3 \cos \left(2x - \frac{\pi}{6}\right) + 7$

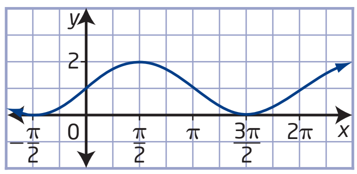
3. **a)** Determine the range of each function.
   i) $y = 3 \cos \left(x - \frac{\pi}{2}\right) + 5$
   ii) $y = -2 \sin (x + \pi) - 3$
   iii) $y = 1.5 \sin x + 4$
   iv) $y = \frac{2}{3} \cos (x + 50°) + \frac{3}{4}$
   b) Describe how to determine the range when given a function of the form $y = a \cos b(x - c) + d$ or $y = a \sin b(x - c) + d$.

4. Match each function with its description in the table.
   a) $y = -2 \cos 2(x + 4) - 1$
   b) $y = 2 \sin 2(x - 4) - 1$
   c) $y = 2 \sin (2x - 4) - 1$
   d) $y = 3 \sin (3x - 9) - 1$
   e) $y = 3 \sin (3x + \pi) - 1$

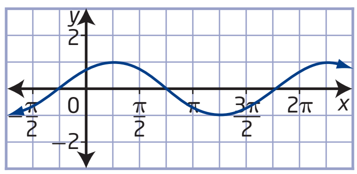
	Amplitude	Period	Phase Shift	Vertical Displacement
A	3	$\frac{2\pi}{3}$	3 right	1 down
B	2	π	2 right	1 down
C	2	π	4 right	1 down
D	2	π	4 left	1 down
E	3	$\frac{2\pi}{3}$	$\frac{\pi}{3}$ left	1 down

5. Match each function with its graph.
   a) $y = \sin \left(x - \frac{\pi}{4}\right)$
   b) $y = \sin \left(x + \frac{\pi}{4}\right)$
   c) $y = \sin x - 1$
   d) $y = \sin x + 1$

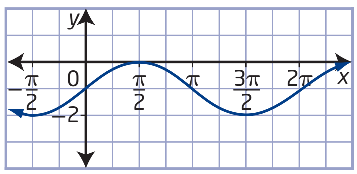
A



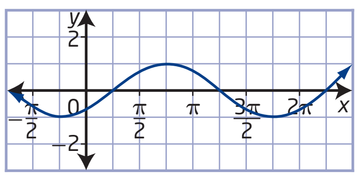
B



C



D



Apply

6. Write the equation of the sine function in the form $y = a \sin b(x - c) + d$ given its characteristics.

a) amplitude 4, period π, phase shift $\frac{\pi}{2}$ to the right, vertical displacement 6 units down

b) amplitude 0.5, period 4π, phase shift $\frac{\pi}{6}$ to the left, vertical displacement 1 unit up

c) amplitude $\frac{3}{4}$, period 720°, no phase shift, vertical displacement 5 units down

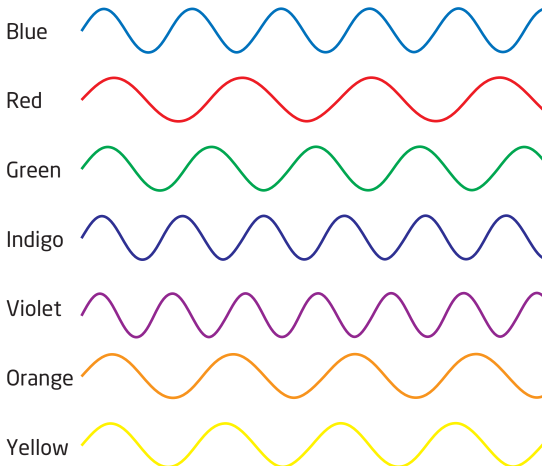
7. The graph of $y = \cos x$ is transformed as described. Determine the values of the parameters a, b, c, and d for the transformed function. Write the equation for the transformed function in the form $y = a \cos b(x - c) + d$.

a) vertical stretch by a factor of 3 about the x-axis, horizontal stretch by a factor of 2 about the y-axis, translated 2 units to the left and 3 units up

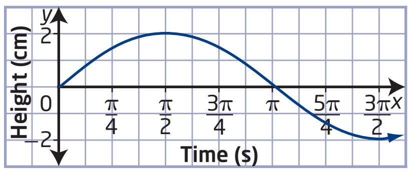
b) vertical stretch by a factor of $\frac{1}{2}$ about the x-axis, horizontal stretch by a factor of $\frac{1}{4}$ about the y-axis, translated 3 units to the right and 5 units down

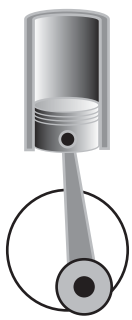
c) vertical stretch by a factor of $\frac{3}{2}$ about the x-axis, horizontal stretch by a factor of 3 about the y-axis, reflected in the x-axis, translated $\frac{\pi}{4}$ units to the right and 1 unit down

8. When white light shines through a prism, the white light is broken into the colours of the visible light spectrum. Each colour corresponds to a different wavelength of the electromagnetic spectrum. Arrange the colours, in order from greatest to smallest period.



9. The piston engine is the most commonly used engine in the world. The height of the piston over time can be modelled by a sine curve. Given the equation for a sine curve, $y = a \sin b(x - c) + d$, which parameter(s) would be affected as the piston moves faster?





10. Victor and Stewart determined the phase shift for the function $f(x) = 4 \sin (2x - 6) + 12$. Victor said that the phase shift was 6 units to the right, while Stewart claimed it was 3 units to the right.

a) Which student was correct? Explain your reasoning.

b) Graph the function to verify your answer from part a).

11. A family of sinusoidal graphs with equations of the form $y = a \sin b(x - c) + d$ is created by changing only the vertical displacement of the function. If the range of the original function is $\{y \mid -3 \le y \le 3, y \in \mathrm{R}\}$, determine the range of the function with each given value of d.

a) $d = 2$

b) $d = -3$

c) $d = -10$

d) $d = 8$

12. Sketch the graph of the curve that results after applying each transformation to the graph of the function $f(x) = \sin x$.

a) $f\left(x - \frac{\pi}{3}\right)$

b) $f\left(x + \frac{\pi}{4}\right)$

c) $f(x) + 3$

d) $f(x) - 4$

13. The range of a trigonometric function in the form $y = a \sin b(x - c) + d$ is $\{y \mid -13 \le y \le 5, y \in \mathrm{R}\}$. State the values of a and d.

14. For each graph of a sinusoidal function, state

i) the amplitude

ii) the period

iii) the phase shift

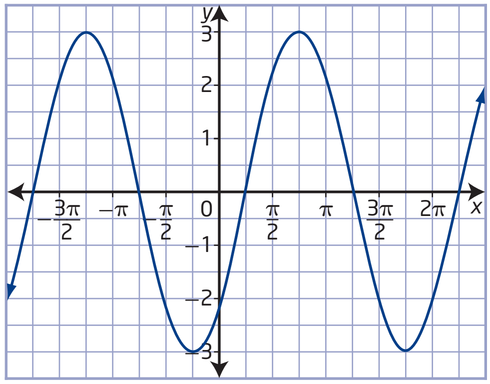
iv) the vertical displacement

v) the domain and range

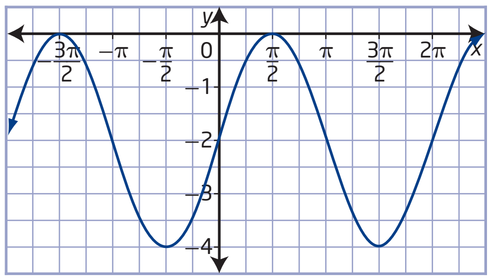
vi) the maximum value of y and the values of x for which it occurs over the interval $0 \le x \le 2\pi$

vii) the minimum value of y and the values of x for which it occurs over the interval $0 \le x \le 2\pi$

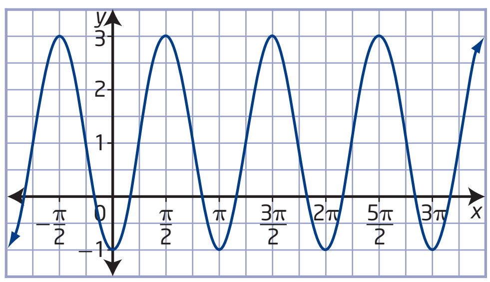
a) a sine function



b) a cosine function

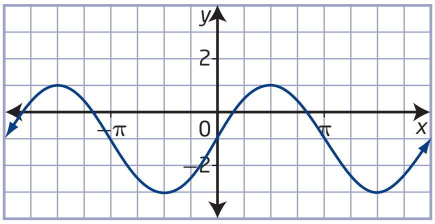


c) a sine function

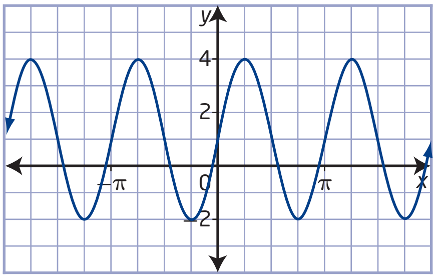


15. Determine an equation in the form $y = a \sin b(x - c) + d$ for each graph.

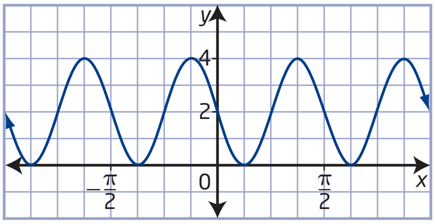
a)



b)

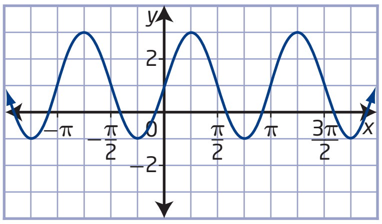


c)

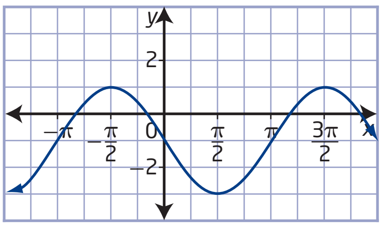


16. For each graph, write an equation in the form $y = a \cos b(x - c) + d$.

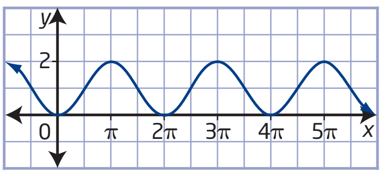
a)



b)



c)



17. a) Graph the function $f(x) = \cos\left(x - \frac{\pi}{2}\right)$.

b) Consider the graph. Write an equation of the function in the form $y = a \sin b(x - c) + d$.

c) What conclusions can you make about the relationship between the two equations of the function?

18. Given the graph of the function $f(x) = \sin x$, what transformation is required so that the function $g(x) = \cos x$ describes the graph of the image function?

19. For each start and end of one cycle of a cosine function in the form $y = 3 \cos b(x - c)$,

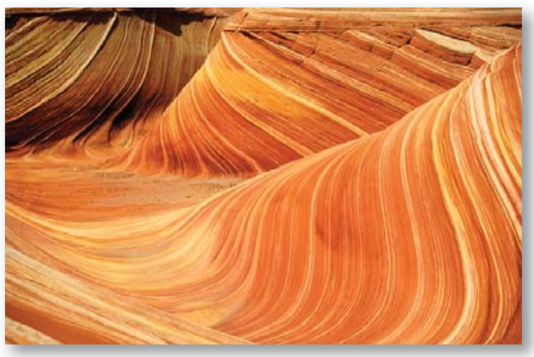
i) state the phase shift, period, and x-intercepts

ii) state the coordinates of the minimum and maximum values

a) $30° \leq x \leq 390°$

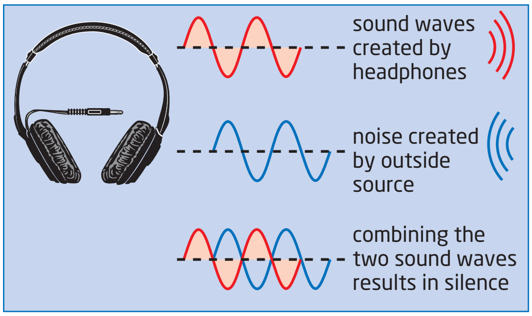
b) $\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$

20. The Wave is a spectacular sandstone formation on the slopes of the Coyote Buttes of the Paria Canyon in Northern Arizona. The Wave is made from 190 million-year-old sand dunes that have turned to red rock. Assume that a cycle of the Wave may be approximated using a cosine curve. The maximum height above sea level is 5100 ft and the minimum height is 5000 ft. The beginning of the cycle is at the 1.75 mile mark of the canyon and the end of this cycle is at the 2.75 mile mark. Write an equation that approximates the pattern of the Wave.



21. Compare the graphs of the functions $y = 3 \sin \frac{\pi}{3}(x - 2) - 1$ and $y = 3 \cos \frac{\pi}{3}\left(x - \frac{7}{2}\right) - 1$. Are the graphs equivalent? Support your answer graphically.

22. Noise-cancelling headphones are designed to give you maximum listening pleasure by cancelling ambient noise and actively creating their own sound waves. These waves mimic the incoming noise in every way, except that they are out of sync with the intruding noise by 180°.



Suppose that the amplitude and period for the sine waves created by the outside noise are 4 and $\frac{\pi}{2}$, respectively. Determine the equation of the sound waves the headphones produce to effectively cancel the ambient noise.

23. The overhang of the roof of a house is designed to shade the windows for cooling in the summer and allow the Sun's rays to enter the house for heating in the winter. The Sun's angle of elevation, A, in degrees, at noon in Estevan, Saskatchewan, can be modelled by the formula $A = -23.5 \sin \frac{360}{365}(x + 102) + 41$, where x is the number of days elapsed beginning with January 1.

a) Use technology to sketch the graph showing the changes in the Sun's angle of elevation throughout the year.

b) Determine the Sun's angle of elevation at noon on February 12.

c) On what date is the angle of elevation the greatest in Estevan?

24. After exercising for 5 min, a person has a respiratory cycle for which the rate of air flow, r, in litres per second, in the lungs is approximated by $r = 1.75 \sin \frac{\pi}{2}t$, where t is the time, in seconds.

a) Determine the time for one full respiratory cycle.

b) Determine the number of cycles per minute.

c) Sketch the graph of the rate of air flow function.

d) Determine the rate of air flow at a time of 30 s. Interpret this answer in the context of the respiratory cycle.

e) Determine the rate of air flow at a time of 7.5 s. Interpret this answer in the context of the respiratory cycle.

Extend

25. The frequency of a wave is the number of cycles that occur in 1 s. Adding two sinusoidal functions with similar, but unequal, frequencies results in a function that pulsates, or exhibits beats. Piano tuners often use this phenomenon to help them tune a piano.

a) Graph the function $y = \cos x + \cos 0.9x$.

b) Determine the amplitude and the period of the resulting wave.

26. **a)** Copy each equation. Fill in the missing values to make the equation true.

i) $4 \sin (x - 30°) = 4 \cos (x - \blacksquare)$

ii) $2 \sin \left(x - \frac{\pi}{4}\right) = 2 \cos (x - \blacksquare)$

iii) $-3 \cos \left(x - \frac{\pi}{2}\right) = 3 \sin (x + \blacksquare)$

iv) $\cos (-2x + 6\pi) = \sin 2(x + \blacksquare)$

b) Choose one of the equations in part a) and explain how you got your answer.

27. Determine the equation of the sine function with

a) amplitude 3, maximum $\left(-\frac{\pi}{2}, 5\right)$, and nearest maximum to the right at $\left(\frac{3\pi}{2}, 5\right)$

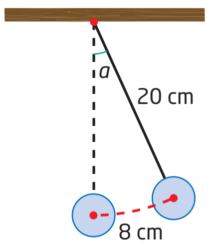
b) amplitude 3, minimum $\left(\frac{\pi}{4}, -2\right)$, and nearest maximum to the right at $\left(\frac{3\pi}{4}, 4\right)$

c) minimum $(-\pi, 3)$ and nearest maximum to the right at $(0, 7)$

d) minimum $(90°, -6)$ and nearest maximum to the right at $(150°, 4)$

28. The angle, P, in radians, between a pendulum and the vertical may be modelled by the equation $P = a \cos bt$, where a represents the maximum angle that the pendulum swings from the vertical; b is the horizontal stretch factor; and t is time, in seconds. The period of a pendulum may be approximated by the formula Period $= 2\pi\sqrt{\frac{L}{g}}$, where L is the pendulum length and g is the acceleration due to gravity (9.8 m/s²).

a) Sketch the graph that models the position of the pendulum in the diagram from $0 \leq t \leq 5$.



b) Determine the position of the pendulum after 6 s. Express your answer to the nearest tenth of a centimetre.

Create Connections

C1 Consider a sinusoidal function of the form $y = a \sin b(x - c) + d$. Describe the effect that each of the parameters a, b, c, and d has on the graph of the function. Compare this to what you learned in Chapter 1 Function Transformations.

C2 Sketch the graphs of $y = -\sin x$ and $y = \sin(-x)$.

a) Compare the two graphs. How are they alike? different?

b) Explain why this happens.

c) How would you expect the graphs of $y = -\cos x$ and $y = \cos(-x)$ to compare?

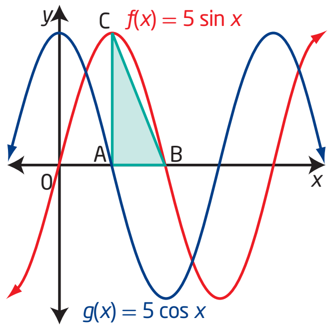
d) Check your hypothesis from part c). If it is incorrect, write a correct statement about the cosine function.

Did You Know?

An *even function* satisfies the property $f(-x) = f(x)$ for all x in the domain of $f(x)$.

An *odd function* satisfies the property $f(-x) = -f(x)$ for all x in the domain of $f(x)$.

C3 Triangle ABC is inscribed between the graphs of $f(x) = 5 \sin x$ and $g(x) = 5 \cos x$. Determine the area of △ABC.



C4 The equation of a sine function can be expressed in the form $y = a \sin b(x - c) + d$. Determine the values of the parameters a, b, c, and/or d, where $a > 0$ and $b > 0$, for each of the following to be true.

a) The period is greater than 2π.

b) The amplitude is greater than 1 unit.

c) The graph passes through the origin.

d) The graph has no x-intercepts.

e) The graph has a y-intercept of a.

f) The length of one cycle is 120°.

5.3

The Tangent Function

Focus on...

- sketching the graph of $y = \tan x$
- determining the amplitude, domain, range, and period of $y = \tan x$
- determining the asymptotes and x-intercepts for the graph of $y = \tan x$
- solving a problem by analysing the graph of the tangent function

You can derive the tangent of an angle from the coordinates of a point on a line tangent to the unit circle at point (1, 0). These values have been tabulated and programmed into scientific calculators and computers. This allows you to apply trigonometry to surveying, engineering, and navigation problems.

Did You Know?

Tangent comes from the Latin word *tangere*, "to touch."

Tangent was first mentioned in 1583 by T. Fincke, who introduced the word *tangens* in Latin. E. Gunter (1624) used the notation *tan*, and J.H. Lambert (1770) discovered the fractional representation of this function.

Investigate the Tangent Function

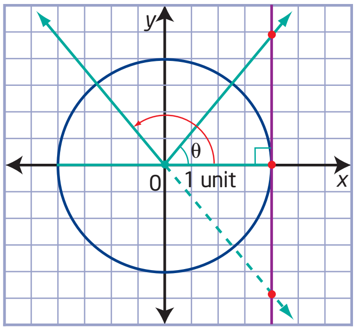
Materials

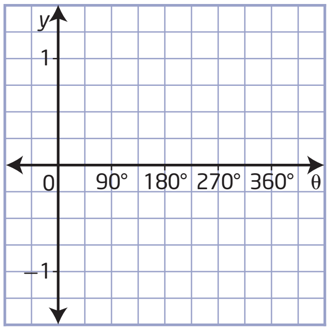
- grid paper
- ruler
- protractor
- compass
- graphing technology

A: Graph the Tangent Function

A tangent line to a curve is a line that touches a curve, or a graph of a function, at a single point.

1. On a piece of grid paper, draw and label the x-axis and y-axis. Draw a circle of radius 1 so that its centre is at the origin. Draw a tangent to the circle at the point where the x-axis intersects the circle on the right side.
2. To sketch the graph of the tangent function over the interval $0° \leq \theta \leq 360°$, you can draw angles in standard position on the unit circle and extend the terminal arm to the right so that it intersects the tangent line, as shown in the diagram. The y-coordinate of the point of intersection represents the value of the tangent function. Plot points represented by the coordinates (angle measure, y-coordinate of point of intersection).





a) Begin with an angle of 0°. Where does the extension of the terminal arm intersect the tangent line?

b) Draw the terminal arm for an angle of 45°. Where does the extension of the terminal arm intersect the tangent line?

c) If the angle is 90°, where does the extension of the terminal arm intersect the tangent line?

What can you conclude about the value of tan 90°? How do you show this on a graph?

d) Use a protractor to measure various angles for the terminal arm. Determine the y-coordinate of the point where the terminal arm intersects the tangent line. Plot the ordered pair (angle measure, y-coordinate on tangent line) on a graph like the one shown above on the right.

Angle Measure	0°	45°	90°	135°	180°	225°	270°	315°	360°
y-coordinate on Tangent Line									

3. Use graphing technology to verify the shape of your graph.

Reflect and Respond

4. When $\theta = 90°$ and $\theta = 270°$, the tangent function is undefined. How does this relate to the graph of the tangent function?

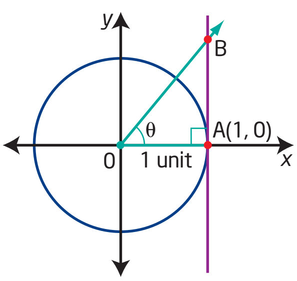
5. What is the period of the tangent function?

6. What is the amplitude of the tangent function? What does this mean?

7. Explain how a point P(x, y) on the unit circle relates to the sine, cosine, and tangent ratios.

B: Connect the Tangent Function to the Slope of the Terminal Arm

8. The diagram shows an angle θ in standard position whose terminal arm intersects the tangent AB at point B. Express the ratio of tan θ in terms of the sides of △AOB.



9. Using your knowledge of special triangles, state the exact value of $\tan 60°$. If $\theta = 60°$ in the diagram, what is the length of line segment AB?
10. Using the measurement of the length of line segment AB from step 9, determine the slope of line segment OB.
11. How does the slope of line segment OB relate to the tangent of an angle in standard position?

Reflect and Respond

12. How could you use the concept of slope to determine the tangent ratio when $\theta = 0°$? when $\theta = 90°$?
13. Using a calculator, determine the values of $\tan \theta$ as θ approaches 90°. What is $\tan 90°$?
14. Explain the relationship between the terminal arm of an angle θ and the tangent of the line passing through the point (1, 0) when $\theta = 90°$. (Hint: Can the terminal arm intersect the tangent line?)

Link the Ideas

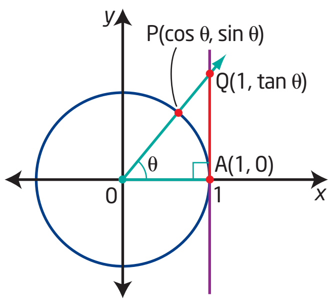
The value of the tangent of an angle θ is the slope of the line passing through the origin and the point on the unit circle $(\cos \theta, \sin \theta)$. You can think of it as the slope of the terminal arm of angle θ in standard position.

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

When $\sin \theta = 0$, what is $\tan \theta$? Explain.
When $\cos \theta = 0$, what is $\tan \theta$? Explain.

The tangent ratio is the length of the line segment tangent to the unit circle at the point A(1, 0) from the x-axis to the terminal arm of angle θ at point Q.

From the diagram, the distance AQ is equal to the y-coordinate of point Q. Therefore, point Q has coordinates $(1, \tan \theta)$.



How could you show that the coordinates of Q are $(1, \tan \theta)$?

Example 1

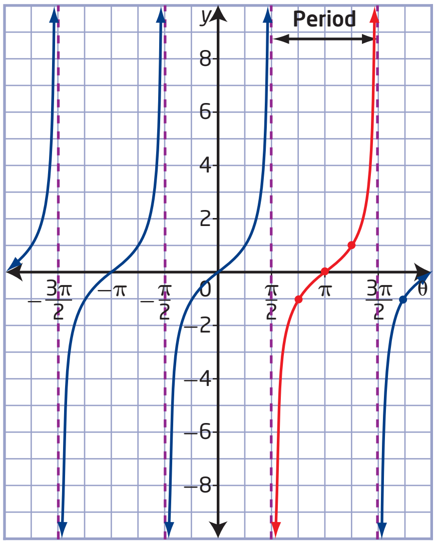
Graph the Tangent Function

Graph the function $y = \tan\theta$ for $-2\pi \le \theta \le 2\pi$. Describe its characteristics.

Solution

The function $y = \tan\theta$ is known as the tangent function. Using the unit circle, you can plot values of y against the corresponding values of θ.

Between asymptotes, the graph of $y = \tan\theta$ passes through a point with y-coordinate -1, a θ-intercept, and a point with y-coordinate 1.



You can observe the properties of the tangent function from the graph.

- The curve is not continuous. It breaks at $\theta = -\frac{3\pi}{2}$, $\theta = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$, and $\theta = \frac{3\pi}{2}$, where the function is undefined.
- $\tan\theta = 0$ when $\theta = -2\pi$, $\theta = -\pi$, $\theta = 0$, $\theta = \pi$, and $\theta = 2\pi$.
- $\tan\theta = 1$ when $\theta = -\frac{7\pi}{4}$, $\theta = -\frac{3\pi}{4}$, $\theta = \frac{\pi}{4}$, and $\theta = \frac{5\pi}{4}$.
- $\tan\theta = -1$ when $\theta = -\frac{5\pi}{4}$, $\theta = -\frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$, and $\theta = \frac{7\pi}{4}$.
- The graph of $y = \tan\theta$ has no amplitude because it has no maximum or minimum values.
- The range of $y = \tan\theta$ is $\{y \mid y \in R\}$.

- As point P moves around the unit circle in either a clockwise or a counterclockwise direction, the tangent curve repeats for every interval of π. The period for $y = \tan\theta$ is π.
- The tangent is undefined whenever $\cos\theta = 0$. This occurs when $\theta = \frac{\pi}{2} + n\pi$, $n \in \mathrm{I}$. At these points, the value of the tangent approaches infinity and is undefined. When graphing the tangent, use dashed lines to show where the value of the tangent is undefined. These vertical lines are called asymptotes.
- The domain of $y = \tan\theta$ is $\left\{\theta \mid \theta \neq \frac{\pi}{2} + n\pi, \theta \in \mathrm{R}, n \in \mathrm{I}\right\}$.

For tangent graphs, the distance between any two consecutive vertical asymptotes represents one complete period.

Why is $\tan\theta$ undefined for $\cos\theta = 0$?

Your Turn

Graph the function $y = \tan\theta$, $0° \leq \theta \leq 360°$. Describe how the characteristics are different from those in Example 1.

Example 2

Model a Problem Using the Tangent Function

A small plane is flying at a constant altitude of 6000 m directly toward an observer. Assume that the ground is flat in the region close to the observer.

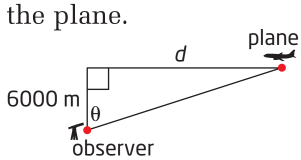
Why is this assumption made?

a) Determine the relation between the horizontal distance, in metres, from the observer to the plane and the angle, in degrees, formed from the vertical to the plane.
b) Sketch the graph of the function.
c) Where are the asymptotes located in this graph? What do they represent?
d) Explain what happens when the angle is equal to 0°.

Solution

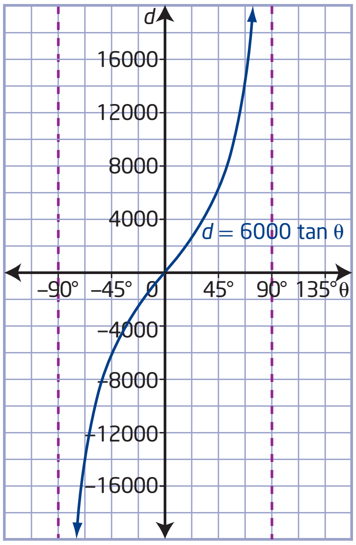
a) Draw a diagram to model the situation.

Let d represent the horizontal distance from the observer to the plane. Let θ represent the angle formed by the vertical and the line of sight to the plane.



$$\tan\theta = \frac{d}{6000}$$
$$d = 6000\tan\theta$$

b) The graph represents the horizontal distance between the plane and the observer. As the plane flies toward the observer, that distance decreases. As the plane moves from directly overhead to the observer's left, the distance values become negative. The domain of the function is $\{\theta \mid -90° < \theta < 90°, \theta \in R\}$.



c) The asymptotes are located at $\theta = 90°$ and $\theta = -90°$. They represent when the plane is on the ground to the right or left of the observer, which is impossible, because the plane is flying in a straight line at a constant altitude of 6000 m.

d) When the angle is equal to 0°, the plane is directly over the head of the observer. The horizontal distance is 0 m.

Your Turn

A small plane is flying at a constant altitude of 5000 m directly toward an observer. Assume the ground is flat in the region close to the observer.

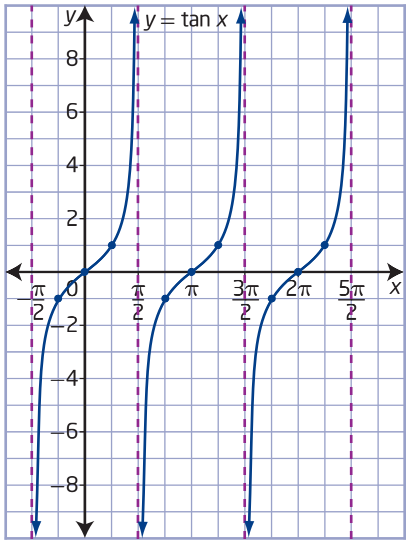
a) Sketch the graph of the function that represents the relation between the horizontal distance, in metres, from the observer to the plane and the angle, in degrees, formed by the vertical and the line of sight to the plane.

b) Use the characteristics of the tangent function to describe what happens to the graph as the plane flies from the right of the observer to the left of the observer.

Key Ideas

- You can use asymptotes and three points to sketch one cycle of a tangent function. To graph $y = \tan x$, draw one asymptote; draw the points where $y = -1$, $y = 0$, and $y = 1$; and then draw another asymptote.

How can you determine the location of the asymptotes for the function $y = \tan x$?



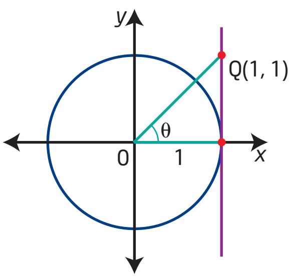
- The tangent function $y = \tan x$ has the following characteristics:
  - The period is π.
  - The graph has no maximum or minimum values.
  - The range is $\{y \mid y \in \mathrm{R}\}$.
  - Vertical asymptotes occur at $x = \frac{\pi}{2} + n\pi$, $n \in \mathrm{I}$.
  - The domain is $\left\{x \mid x \neq \frac{\pi}{2} + n\pi, x \in \mathrm{R}, n \in \mathrm{I}\right\}$.
  - The x-intercepts occur at $x = n\pi$, $n \in \mathrm{I}$.
  - The y-intercept is 0.

Check Your Understanding

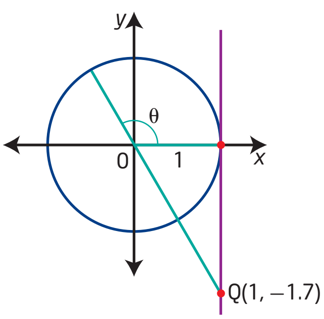
Practise

1. For each diagram, determine $\tan \theta$ and the value of θ, in degrees. Express your answer to the nearest tenth, when necessary.

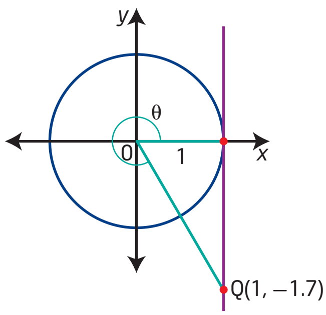
a)



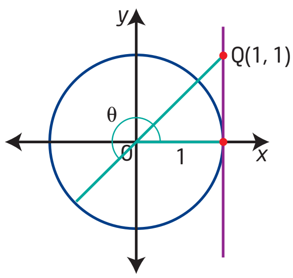
b)



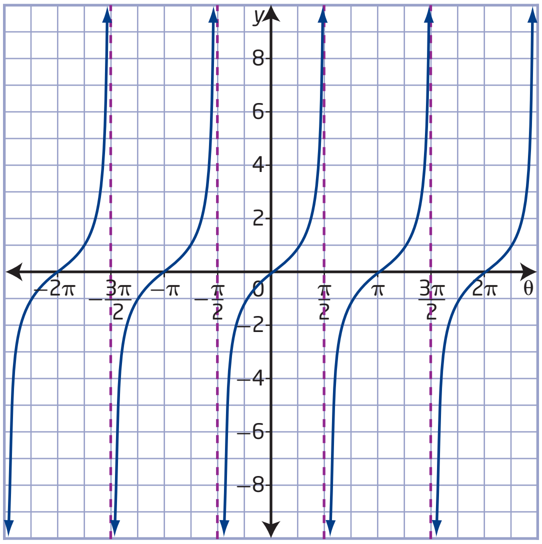
c)



d)



2. Use the graph of the function $y = \tan \theta$ to determine each value.



a) $\tan \frac{\pi}{2}$

b) $\tan \frac{3\pi}{4}$

c) $\tan \left(-\frac{7\pi}{4}\right)$

d) $\tan 0$

e) $\tan \pi$

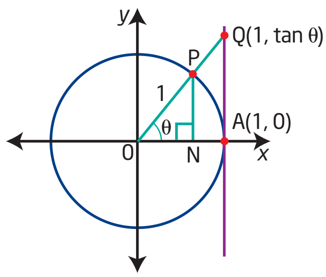
f) $\tan \frac{5\pi}{4}$

3. Does $y = \tan x$ have an amplitude? Explain.

4. Use graphing technology to graph $y = \tan x$ using the following window settings: x: [−360°, 360°, 30°] and y: [−3, 3, 1]. Trace along the graph to locate the value of $\tan x$ when $x = 60°$. Predict the other values of x that will produce the same value for $\tan x$ within the given domain. Verify your predictions.

Apply

5. In the diagram, $\triangle$PON and $\triangle$QOA are similar triangles. Use the diagram to justify the statement $\tan \theta = \frac{\sin \theta}{\cos \theta}$.



6. Point P(x, y) is plotted where the terminal arm of angle θ intersects the unit circle.

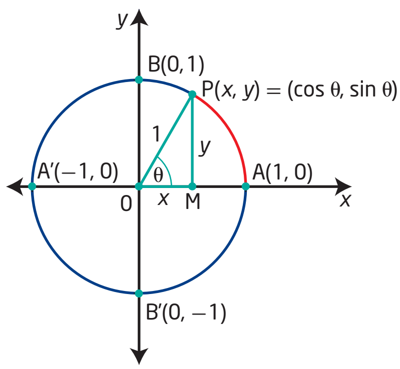
a) Use P(x, y) to determine the slope of the terminal arm.

b) Explain how your result from part a) is related to $\tan \theta$.

c) Write your results for the slope from part a) in terms of sine and cosine.

d) From your answer in part c), explain how you could determine $\tan \theta$ when the coordinates of point P are known.

7. Consider the unit circle shown.



a) From $\triangle$POM, write the ratio for $\tan \theta$.

b) Use $\cos \theta$ and $\sin \theta$ to write the ratio for $\tan \theta$.

c) Explain how your answers from parts a) and b) are related.

8. The graph of $y = \tan \theta$ appears to be vertical as θ approaches 90°.

a) Copy and complete the table. Use a calculator to record the tangent values as θ approaches 90°.

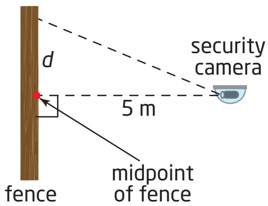
θ	$\tan \theta$
89.5°	
89.9°	
89.999°	
89.999 999°	

b) What happens to the value of $\tan \theta$ as θ approaches 90°?

c) Predict what will happen as θ approaches 90° from the other direction.

θ	$\tan \theta$
90.5°	
90.01°	
90.001°	
90.000 001°	

9. A security camera scans a long straight fence that encloses a section of a military base. The camera is mounted on a post that is located 5 m from the midpoint of the fence. The camera makes one complete rotation in 60 s.



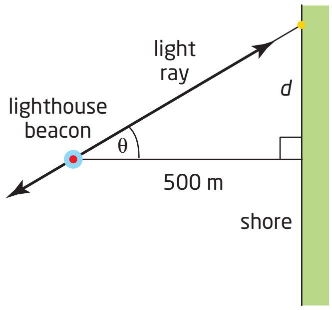
a) Determine the tangent function that represents the distance, d, in metres, along the fence from its midpoint as a function of time, t, in seconds, if the camera is aimed at the midpoint of the fence at $t = 0$.

b) Graph the function in the interval $-15 \leq t \leq 15$.

c) What is the distance from the midpoint of the fence at $t = 10$ s, to the nearest tenth of a metre?

d) Describe what happens when $t = 15$ s.

10. A rotating light on top of a lighthouse sends out rays of light in opposite directions. As the beacon rotates, the ray at angle θ makes a spot of light that moves along the shore. The lighthouse is located 500 m from the shoreline and makes one complete rotation every 2 min.



a) Determine the equation that expresses the distance, d, in metres, as a function of time, t, in minutes.

b) Graph the function in part a).

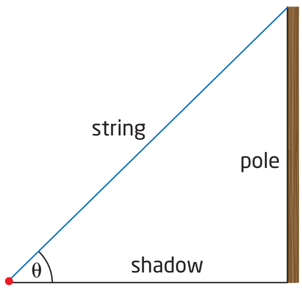
c) Explain the significance of the asymptote in the graph at $\theta = 90°$.

Did You Know?

The Fisgard Lighthouse was the first lighthouse built on Canada's west coast. It was built in 1860 before Vancouver Island became part of Canada and is located at the entrance to Esquimalt harbour.

11. A plane flying at an altitude of 10 km over level ground will pass directly over a radar station. Let d be the ground distance from the antenna to a point directly under the plane. Let x represent the angle formed from the vertical at the radar station to the plane. Write d as a function of x and graph the function over the interval $0 \leq x \leq \frac{\pi}{2}$.

12. Andrea uses a pole of known height, a piece of string, a measuring tape, and a calculator for an assignment. She places the pole in a vertical position in the school field and runs the string from the top of the pole to the tip of the shadow formed by the pole. Every 15 min, Andrea measures the length of the shadow and then calculates the slope of the string and the measure of the angle. She records the data and graphs the slope as a function of the angle.



a) What type of graph would you expect Andrea to graph to represent her data?

b) When the Sun is directly overhead and no shadow results, state the slope of the string. How does Andrea's graph represent this situation?

Extend

13. a) Graph the line $y = \frac{3}{4}x$, where $x > 0$. Mark an angle θ that represents the angle formed by the line and the positive x-axis. Plot a point with integral coordinates on the line $y = \frac{3}{4}x$.

b) Use these coordinates to determine $\tan \theta$.

c) Compare the equation of the line with your results in part b). Make a conjecture based on your findings.

14. Have you ever wondered how a calculator or computer program evaluates the sine, cosine, or tangent of a given angle? The calculator or computer program approximates these values using a power series. The terms of a power series contain ascending positive integral powers of a variable. The more terms in the series, the more accurate the approximation. With a calculator in radian mode, verify the following for small values of x, for example, $x = 0.5$.

a) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}$

b) $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$

c) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$

Create Connections

C1 How does the domain of $y = \tan x$ differ from that of $y = \sin x$ and $y = \cos x$? Explain why.

C2 a) On the same set of axes, graph the functions $f(x) = \cos x$ and $g(x) = \tan x$. Describe how the two functions are related.

b) On the same set of axes, graph the functions $f(x) = \sin x$ and $g(x) = \tan x$. Describe how the two functions are related.

C3 Explain how the equation $\tan (x + \pi) = \tan x$ relates to circular functions.

5.4

Equations and Graphs of Trigonometric Functions

Focus on . . .

- using the graphs of trigonometric functions to solve equations
- analysing a trigonometric function to solve a problem
- determining a trigonometric function that models a problem
- using a model of a trigonometric function for a real-world situation

One of the most useful characteristics of trigonometric functions is their periodicity. For example, the times of sunsets, sunrises, and comet appearances; seasonal temperature changes; the movement of waves in the ocean; and even the quality of a musical sound can be described using trigonometric functions. Mathematicians and scientists use the periodic nature of trigonometric functions to develop mathematical models to predict many natural phenomena.

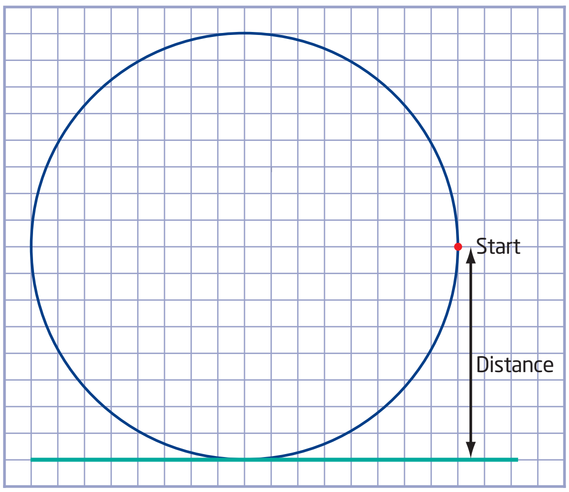
Investigate Trigonometric Equations

Materials

- marker
- ruler
- compass
- stop watch
- centimetre grid paper

Work with a partner.

1. On a sheet of centimetre grid paper, draw a circle of radius 8 cm. Draw a line tangent to the bottom of the circle.



2. Place a marker at the three o'clock position on the circle. Move the marker around the circle in a counterclockwise direction, measuring the time it takes to make one complete trip around the circle.
3. Move the marker around the circle a second time stopping at time intervals of 2 s. Measure the vertical distance from the marker to the tangent line. Complete a table of times and distances.

Time (s)	0	2	4	6	8	10	12	14	16	18	20
Distance (cm)	8										

4. Create a scatterplot of distance versus time. Draw a smooth curve connecting the points.
5. Write a function for the resulting curve.
6. a) From your initial starting position, move the marker around the circle in a counterclockwise direction for 3 s. Measure the vertical distance of the marker from the tangent line. Label this point on your graph.
   b) Continue to move the marker around the circle to a point that is the same distance as the distance you recorded in part a). Label this point on your graph.
   c) How do these two points relate to your function in step 5?
   d) How do the measured and calculated distances compare?
7. Repeat step 6 for other positions on the circle.

Aim to complete one revolution in 20 s. You may have to practice this several times to maintain a consistent speed.

Did You Know?

A scatter plot is the result of plotting data that can be represented as ordered pairs on a graph.

Reflect and Respond

8. What is the connection between the circular pattern followed by your marker and the graph of distance versus time?
9. Describe how the circle, the graph, and the function are related.

Link the Ideas

You can represent phenomena with periodic behaviour or wave characteristics by trigonometric functions or model them approximately with sinusoidal functions. You can identify a trend or pattern, determine an appropriate mathematical model to describe the process, and use it to make predictions (interpolate or extrapolate).

You can use graphs of trigonometric functions to solve trigonometric equations that model periodic phenomena, such as the swing of a pendulum, the motion of a piston in an engine, the motion of a Ferris wheel, variations in blood pressure, the hours of daylight throughout a year, and vibrations that create sounds.

Example 1

Solve a Trigonometric Equation in Degrees

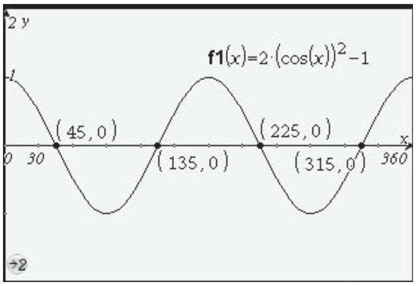
Determine the solutions for the trigonometric equation $2\cos^2 x - 1 = 0$ for the interval $0° \le x \le 360°$.

Solution

Method 1: Solve Graphically

Graph the related function $f(x) = 2\cos^2 x - 1$.

Use the graphing window [0, 360, 30] by [−2, 2, 1].



The solutions to the equation $2\cos^2 x - 1 = 0$ for the interval $0° \le x \le 360°$ are the x-intercepts of the graph of the related function.

The solutions for the interval $0° \le x \le 360°$ are $x = 45°, 135°, 225°$, and $315°$.

Method 2: Solve Algebraically

$$\begin{aligned} 2\cos^2 x - 1 &= 0 \\ 2\cos^2 x &= 1 \\ \cos^2 x &= \frac{1}{2} \\ \cos x &= \pm\sqrt{\frac{1}{2}} \end{aligned}$$

Why is the ± symbol used?

For $\cos x = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$, the angles in the interval $0° \le x \le 360°$ that satisfy the equation are $45°$ and $315°$.

For $\cos x = -\sqrt{\frac{1}{2}}$, the angles in the interval $0° \le x \le 360°$ that satisfy the equation are $135°$ and $225°$.

The solutions for the interval $0° \le x \le 360°$ are $x = 45°, 135°, 225°$, and $315°$.

Your Turn

Determine the solutions for the trigonometric equation $4\sin^2 x - 3 = 0$ for the interval $0° \le x \le 360°$.

Example 2

Solve a Trigonometric Equation in Radians

Determine the general solutions for the trigonometric equation $16 = 6 \cos \frac{\pi}{6}x + 14$. Express your answers to the nearest hundredth.

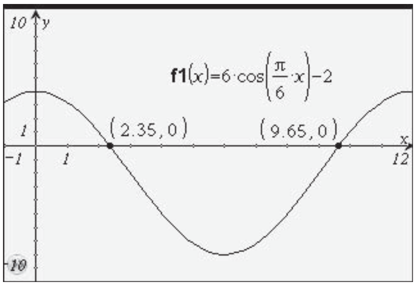
Solution

Method 1: Determine the Zeros of the Function

Rearrange the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ so that one side is equal to 0.

$6 \cos \frac{\pi}{6}x - 2 = 0$

Graph the related function $y = 6 \cos \frac{\pi}{6}x - 2$. Use the window $[-1, 12, 1]$ by $[-10, 10, 1]$.



Why should you set the calculator to radian mode?

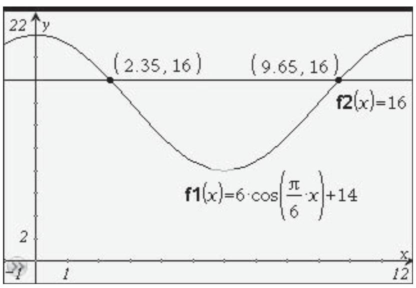
The solutions to the equation $6 \cos \frac{\pi}{6}x - 2 = 0$ are the x-intercepts.

The x-intercepts are approximately $x = 2.35$ and $x = 9.65$. The period of the function is 12 radians. So, the x-intercepts repeat in multiples of 12 radians from each of the original intercepts.

The general solutions to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ are $x \approx 2.35 + 12n$ radians and $x \approx 9.65 + 12n$ radians, where n is an integer.

Method 2: Determine the Points of Intersection

Graph the functions $y = 6 \cos \frac{\pi}{6}x + 14$ and $y = 16$ using a window $[-1, 12, 1]$ by $[-2, 22, 2]$.



The solution to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ is given by the points of intersection of the curve $y = 6 \cos \frac{\pi}{6}x + 14$ and the line $y = 16$. In the interval $0 \le x \le 12$, the points of intersection occur at $x \approx 2.35$ and $x \approx 9.65$.

The period of the function is 12 radians. The points of intersection repeat in multiples of 12 radians from each of the original intercepts.

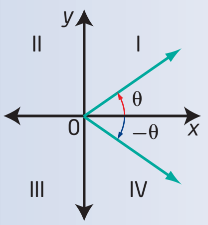
The general solutions to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ are $x \approx 2.35 + 12n$ radians and $x \approx 9.65 + 12n$ radians, where n is an integer.

Method 3: Solve Algebraically

$$\begin{aligned}
16 &= 6 \cos \frac{\pi}{6}x + 14 \\
2 &= 6 \cos \frac{\pi}{6}x \\
\frac{2}{6} &= \cos \frac{\pi}{6}x \\
\frac{1}{3} &= \cos \frac{\pi}{6}x \\
\cos^{-1}\left(\frac{1}{3}\right) &= \frac{\pi}{6}x \\
1.2309\ldots &= \frac{\pi}{6}x \\
x &= 2.3509\ldots
\end{aligned}$$

Did You Know?

No matter in which quadrant θ falls, $-\theta$ has the same reference angle and both θ and $-\theta$ are located on the same side of the y-axis. Since $\cos \theta$ is positive on the right side of the y-axis and negative on the left side of the y-axis, $\cos \theta = \cos (-\theta)$.



Since the cosine function is positive in quadrants I and IV, a second possible value of x can be determined. In quadrant IV, the angle is $2\pi - \frac{\pi}{6}x$.

$$\begin{aligned}
\frac{1}{3} &= \cos \left(2\pi - \frac{\pi}{6}x\right) \\
\cos^{-1}\left(\frac{1}{3}\right) &= 2\pi - \frac{\pi}{6}x \\
\frac{\pi}{6}x &= 2\pi - \cos^{-1}\left(\frac{1}{3}\right) \\
x &= 12 - \frac{6}{\pi} \cos^{-1}\left(\frac{1}{3}\right) \\
x &= 9.6490\ldots
\end{aligned}$$

Two solutions to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ are $x \approx 2.35$ and $x \approx 9.65$.

The period of the function is 12 radians, then the solutions repeat in multiples of 12 radians from each original solution.

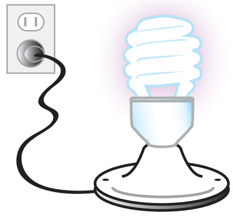
The general solutions to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ are $x \approx 2.35 + 12n$ radians and $x \approx 9.65 + 12n$ radians, where n is an integer.

Your Turn

Determine the general solutions for the trigonometric equation $10 = 6 \sin \frac{\pi}{4}x + 8$.

Example 3

Model Electric Power



The electricity coming from power plants into your house is alternating current (AC). This means that the direction of current flowing in a circuit is constantly switching back and forth. In Canada, the current makes 60 complete cycles each second.

The voltage can be modelled as a function of time using the sine function $V = 170 \sin 120\pi t$.

a) What is the period of the current in Canada?

b) Graph the voltage function over two cycles. Explain what the scales on the axes represent.

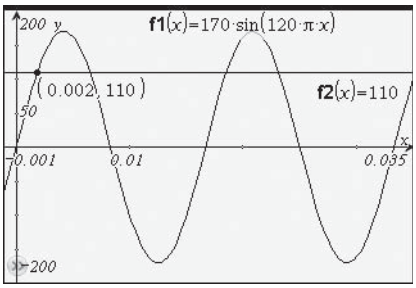
c) Suppose you want to switch on a heat lamp for an outdoor patio. If the heat lamp requires 110 V to start up, determine the time required for the voltage to first reach 110 V.

Solution

a) Since there are 60 complete cycles in each second, each cycle takes $\frac{1}{60}$ s. So, the period is $\frac{1}{60}$.

b) To graph the voltage function over two cycles on a graphing calculator, use the following window settings:

x: [−0.001, 0.035, 0.01]
y: [−200, 200, 50]



The y-axis represents the number of volts. Each tick mark on the y-axis represents 50 V.

The x-axis represents the time passed. Each tick mark on the x-axis represents 0.01 s.

c) Graph the line $y = 110$ and determine the first point of intersection with the voltage function. It will take approximately 0.002 s for the voltage to first reach 110 V.

Your Turn

In some Caribbean countries, the current makes 50 complete cycles each second and the voltage is modelled by $V = 170 \sin 100\pi t$.

a) Graph the voltage function over two cycles. Explain what the scales on the axes represent.

b) What is the period of the current in these countries?

c) How many times does the voltage reach 110 V in the first second?

Did You Know?

Tidal power is a form of hydroelectric power that converts the energy of tides into electricity. Estimates of Canada's tidal energy potential off the Canadian Pacific coast are equivalent to approximately half of the country's current electricity demands.

Did You Know?

The number of cycles per second of a periodic phenomenon is called the frequency. The hertz (Hz) is the SI unit of frequency. In Canada, the frequency standard for AC is 60 Hz.

Voltages are expressed as root mean square (RMS) voltage. RMS is the square root of the mean of the squares of the values. The RMS voltage is given by $\frac{\text{peak voltage}}{\sqrt{2}}$. What is the RMS voltage for Canada?

Example 4

Model Hours of Daylight

Iqaluit is the territorial capital and the largest community of Nunavut. Iqaluit is located at latitude 63° N. The table shows the number of hours of daylight on the 21st day of each month as the day of the year on which it occurs for the capital (based on a 365-day year).

Why is the 21st day of each month chosen for the data in the table?

Hours of Daylight by Day of the Year for Iqaluit, Nunavut

Jan 21	Feb 21	Mar 21	Apr 21	May 21	June 21	July 21	Aug 21	Sept 21	Oct 21	Nov 21	Dec 21
21	52	80	111	141	172	202	233	264	294	325	355
6.12	9.36	12.36	15.69	18.88	20.83	18.95	15.69	12.41	9.24	6.05	4.34

a) Draw a scatter plot for the number of hours of daylight, h, in Iqaluit on the day of the year, t.

b) Which sinusoidal function will best fit the data without requiring a phase shift: $h(t) = \sin t$, $h(t) = -\sin t$, $h(t) = \cos t$, or $h(t) = -\cos t$? Explain.

c) Write the sinusoidal function that models the number of hours of daylight.

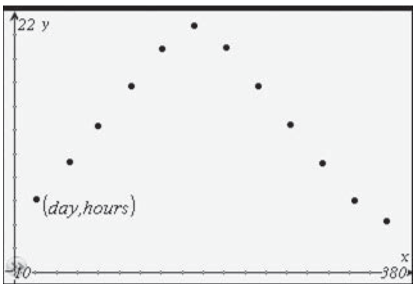
d) Graph the function from part c).

e) Estimate the number of hours of daylight on each date.

i) March 15 (day 74) **ii)** July 10 (day 191) **iii)** December 5 (day 339)

Solution

a) Graph the data as a scatter plot.



b) Note that the data starts at a minimum value, climb to a maximum value, and then decrease to the minimum value. The function $h(t) = -\cos t$ exhibits this same behaviour.

c) The maximum value is 20.83, and the minimum value is 4.34. Use these values to find the amplitude and the equation of the sinusoidal axis.

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$

$$|a| = \frac{20.83 - 4.34}{2}$$

$$|a| = 8.245$$

The sinusoidal axis lies halfway between the maximum and minimum values. Its position will determine the value of d.

$$d = \frac{\text{maximum value} + \text{minimum value}}{2}$$

$$d = \frac{20.83 + 4.34}{2}$$

$$d = 12.585$$

Determine the value of b. You know that the period is 365 days.

$$\text{Period} = \frac{2\pi}{|b|}$$

$$365 = \frac{2\pi}{|b|}$$

$$b = \frac{2\pi}{365}$$

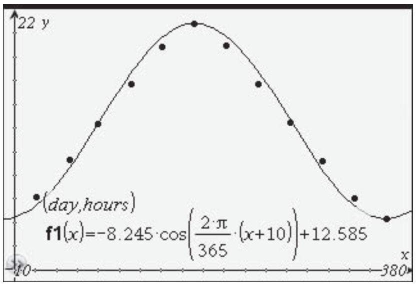
Choose b to be positive.

Why is the period 365 days?

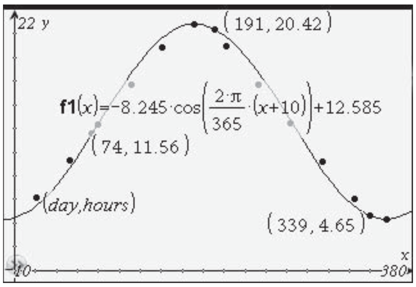
Determine the phase shift, the value of c. For $h(t) = -\cos t$ the minimum value occurs at $t = 0$. For the daylight hours curve, the actual minimum occurs at day 355, which represents a 10-day shift to the left. Therefore, $c = -10$.

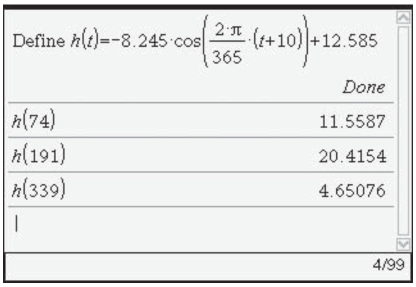
The number of hours of daylight, h, on the day of the year, t, is given by the function $h(t) = -8.245 \cos\left(\frac{2\pi}{365}(t + 10)\right) + 12.585$.

d) Graph the function in the same window as your scatter plot.



e) Use the value feature of the calculator or substitute the values into the equation of the function.





i) The number of hours of daylight on March 15 (day 74) is approximately 11.56 h.

ii) The number of hours of daylight on July 10 (day 191) is approximately 20.42 h.

iii) The number of hours of daylight on December 5 (day 339) is approximately 4.65 h.

Your Turn

Windsor, Ontario, is located at latitude 42° N. The table shows the number of hours of daylight on the 21st day of each month as the day of the year on which it occurs for this city.

Hours of Daylight by Day of the Year for Windsor, Ontario											
21	52	80	111	141	172	202	233	264	294	325	355
9.62	10.87	12.20	13.64	14.79	15.28	14.81	13.64	12.22	10.82	9.59	9.08

a) Draw a scatter plot for the number of hours of daylight, h, in Windsor, Ontario on the day of the year, t.

b) Write the sinusoidal function that models the number of hours of daylight.

c) Graph the function from part b).

d) Estimate the number of hours of daylight on each date.

i) March 10

ii) July 24

iii) December 3

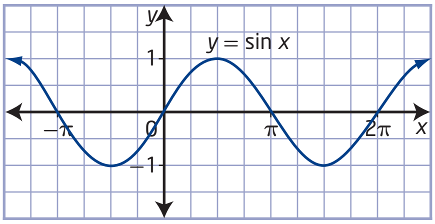
e) Compare the graphs for Iqaluit and Windsor. What conclusions can you draw about the number of hours of daylight for the two locations?

Key Ideas

- You can use sinusoidal functions to model periodic phenomena that do not involve angles as the independent variable.
- You can adjust the amplitude, phase shift, period, and vertical displacement of the basic trigonometric functions to fit the characteristics of the real-world application being modelled.
- You can use technology to create the graph modelling the application. Use this graph to interpolate or extrapolate information required to solve the problem.
- You can solve trigonometric equations graphically. Use the graph of a function to determine the x-intercepts or the points of intersection with a given line. You can express your solutions over a specified interval or as a general solution.

Practise

1. a) Use the graph of $y = \sin x$ to determine the solutions to the equation $\sin x = 0$ for the interval $0 \le x \le 2\pi$.

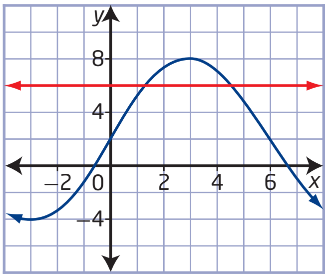


b) Determine the general solution for $\sin x = 0$.

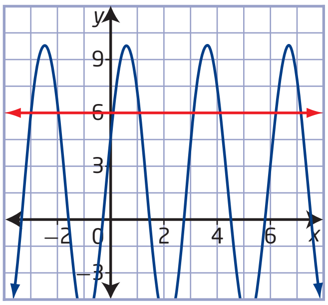
c) Determine the solutions for $\sin 3x = 0$ in the interval $0 \le x \le 2\pi$.

2. The partial sinusoidal graphs shown below are intersected by the line $y = 6$. Each point of intersection corresponds to a value of x where $y = 6$. For each graph shown determine the approximate value of x where $y = 6$.

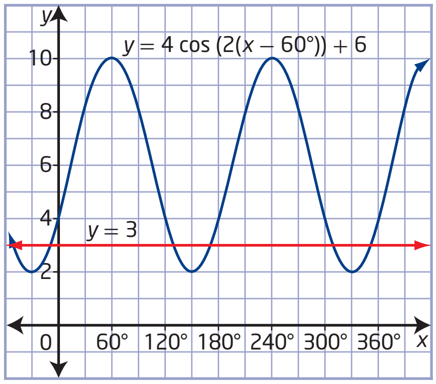
a)



b)



3. The partial graph of a sinusoidal function $y = 4 \cos (2(x - 60°)) + 6$ and the line $y = 3$ are shown below. From the graph determine the approximate solutions to the equation $4 \cos (2(x - 60°)) + 6 = 3$.



4. Solve each of the following equations graphically.

a) $-2.8 \sin \left(\frac{\pi}{6}(x - 12)\right) + 16 = 16$, $0 \le x \le 2\pi$

b) $12 \cos (2(x - 45°)) + 8 = 10$, $0° \le x \le 360°$

c) $7 \cos (3x - 18) = 4$, $0 \le x \le 2\pi$

d) $6.2 \sin (4(x + 8°)) - 1 = 4$, $0° \le x \le 360°$

5. Solve each of the following equations.

a) $\sin \left(\frac{\pi}{4}(x - 6)\right) = 0.5$, $0 \le x \le 2\pi$

b) $4 \cos (x - 45°) + 7 = 10$, $0° \le x \le 360°$

c) $8 \cos (2x - 5) = 3$, general solution in radians

d) $5.2 \sin (45(x + 8°)) - 1 = -3$, general solution in degrees

6. State a possible domain and range for the given functions, which represent real-world applications.

a) The population of a lakeside town with large numbers of seasonal residents is modelled by the function $P(t) = 6000 \sin (t - 8) + 8000$.

b) The height of the tide on a given day can be modelled using the function $h(t) = 6 \sin (t - 5) + 7$.

c) The height above the ground of a rider on a Ferris wheel can be modelled by $h(t) = 6 \sin 3(t - 30) + 12$.

d) The average daily temperature may be modelled by the function $h(t) = 9 \cos \frac{2\pi}{365}(t - 200) + 14$.

7. A trick from Victorian times was to listen to the pitch of a fly's buzz, reproduce the musical note on the piano, and say how many times the fly's wings had flapped in 1 s. If the fly's wings flap 200 times in one second, determine the period of the musical note.

8. Determine the period, the sinusoidal axis, and the amplitude for each of the following.

a) The first maximum of a sine function occurs at the point (30°, 24), and the first minimum to the right of the maximum occurs at the point (80°, 6).

b) The first maximum of a cosine function occurs at (0, 4), and the first minimum to the right of the maximum occurs at $\left(\frac{2\pi}{3}, -16\right)$.

c) An electron oscillates back and forth 50 times per second, and the maximum and minimum values occur at +10 and −10, respectively.

Apply

9. A point on an industrial flywheel experiences a motion described by the function $h(t) = 13 \cos \left(\frac{2\pi}{0.7}t\right) + 15$, where h is the height, in metres, and t is the time, in minutes.

a) What is the maximum height of the point?

b) After how many minutes is the maximum height reached?

c) What is the minimum height of the point?

d) After how many minutes is the minimum height reached?

e) For how long, within one cycle, is the point less than 6 m above the ground?

f) Determine the height of the point if the wheel is allowed to turn for 1 h 12 min.

10. Michelle is balancing the wheel on her bicycle. She has marked a point on the tire that when rotated can be modelled by the function $h(t) = 59 + 24 \sin 125t$, where h is the height, in centimetres, and t is the time, in seconds. Determine the height of the mark, to the nearest tenth of a centimetre, when $t = 17.5$ s.

11. The typical voltage, V, in volts (V), supplied by an electrical outlet in Cuba is a sinusoidal function that oscillates between −155 V and +155 V and makes 60 complete cycles each second. Determine an equation for the voltage as a function of time, t.

12. The University of Calgary's Institute for Space Research is leading a project to launch Cassiope, a hybrid space satellite. Cassiope will follow a path that may be modelled by the function $h(t) = 350 \sin 28\pi(t - 25) + 400$, where h is the height, in kilometres, of the satellite above Earth and t is the time, in days.

a) Determine the period of the satellite.

b) How many minutes will it take the satellite to orbit Earth?

c) How many orbits per day will the satellite make?

13. The Arctic fox is common throughout the Arctic tundra. Suppose the population, F, of foxes in a region of northern Manitoba is modelled by the function $F(t) = 500 \sin \frac{\pi}{12}t + 1000$, where t is the time, in months.

a) How many months would it take for the fox population to drop to 650? Round your answer to the nearest month.

b) One of the main food sources for the Arctic fox is the lemming. Suppose the population, L, of lemmings in the region is modelled by the function $L(t) = 5000 \sin \frac{\pi}{12}(t - 12) + 10\,000$. Graph the function $L(t)$ using the same set of axes as for $F(t)$.

c) From the graph, determine the maximum and minimum numbers of foxes and lemmings and the months in which these occur.

d) Describe the relationships between the maximum, minimum, and mean points of the two curves in terms of the lifestyles of the foxes and lemmings. List possible causes for the fluctuation in populations.

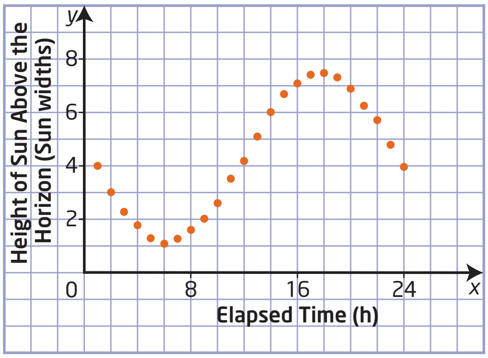
14. Office towers are designed to sway with the wind blowing from a particular direction. In one situation, the horizontal sway, h, in centimetres, from vertical can be approximated by the function $h = 40 \sin 0.526t$, where t is the time, in seconds.

a) Graph the function using graphing technology. Use the following window settings: x: [0, 12, 1], y: [−40, 40, 5].

b) If a guest arrives on the top floor at $t = 0$, how far will the guest have swayed from the vertical after 2.034 s?

c) If a guest arrives on the top floor at $t = 0$, how many seconds will have elapsed before the guest has swayed 20 cm from the vertical?

15. In Inuvik, Northwest Territories (latitude 68.3° N), the Sun does not set for 56 days during the summer. The midnight Sun sequence below illustrates the rise and fall of the polar Sun during a day in the summer.



a) Determine the maximum and minimum heights of the Sun above the horizon in terms of Sun widths.

b) What is the period?

c) Determine the sinusoidal equation that models the midnight Sun.

Did You Know?

In 2010, a study showed that the Sun's width, or diameter, is a steady 1 500 000 km. The researchers discovered over a 12-year period that the diameter changed by less than 1 km.

16. The table shows the average monthly temperature in Winnipeg, Manitoba, in degrees Celsius.

Average Monthly Temperatures for Winnipeg, Manitoba (°C)					
Jan	Feb	Mar	Apr	May	Jun
−16.5	−12.7	−5.6	3	11.3	17.3

Average Monthly Temperatures for Winnipeg, Manitoba (°C)					
Jul	Aug	Sep	Oct	Nov	Dec
19.7	18	12.5	4.5	−4.3	−11.7

a) Plot the data on a scatter plot.

b) Determine the temperature that is halfway between the maximum average monthly temperature and the minimum average monthly temperature for Winnipeg.

c) Determine a sinusoidal function to model the temperature for Winnipeg.

d) Graph your model. How well does your model fit the data?

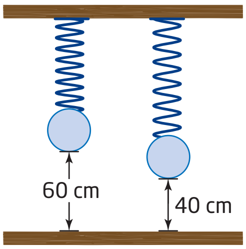
e) For how long in a 12-month period does Winnipeg have a temperature greater than or equal to 16 °C?

17. An electric heater turns on and off on a cyclic basis as it heats the water in a hot tub. The water temperature, T, in degrees Celsius, varies sinusoidally with time, t, in minutes. The heater turns on when the temperature of the water reaches 34 °C and turns off when the water temperature is 43 °C. Suppose the water temperature drops to 34 °C and the heater turns on. After another 30 min the heater turns off, and then after another 30 min the heater starts again.

a) Write the equation that expresses temperature as a function of time.

b) Determine the temperature 10 min after the heater first turns on.

18. A mass attached to the end of a long spring is bouncing up and down. As it bounces, its distance from the floor varies sinusoidally with time. When the mass is released, it takes 0.3 s to reach a high point of 60 cm above the floor. It takes 1.8 s for the mass to reach the first low point of 40 cm above the floor.



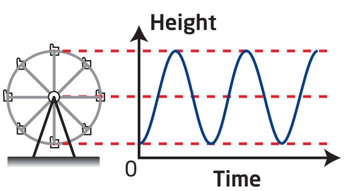
a) Sketch the graph of this sinusoidal function.

b) Determine the equation for the distance from the floor as a function of time.

c) What is the distance from the floor when the stopwatch reads 17.2 s?

d) What is the first positive value of time when the mass is 59 cm above the floor?

19. A Ferris wheel with a radius of 10 m rotates once every 60 s. Passengers get on board at a point 2 m above the ground at the bottom of the Ferris wheel. A sketch for the first 150 s is shown.



a) Write an equation to model the path of a passenger on the Ferris wheel, where the height is a function of time.

b) If Emily is at the bottom of the Ferris wheel when it begins to move, determine her height above the ground, to the nearest tenth of a metre, when the wheel has been in motion for 2.3 min.

c) Determine the amount of time that passes before a rider reaches a height of 18 m for the first time. Determine one other time the rider will be at that height within the first cycle.

20. The Canadian National Historic Windpower Centre, at Etzikom, Alberta, has various styles of windmills on display. The tip of the blade of one windmill reaches its minimum height of 8 m above the ground at a time of 2 s. Its maximum height is 22 m above the ground. The tip of the blade rotates 12 times per minute.

a) Write a sine or a cosine function to model the rotation of the tip of the blade.

b) What is the height of the tip of the blade after 4 s?

c) For how long is the tip of the blade above a height of 17 m in the first 10 s?

21. In a 366-day year, the average daily maximum temperature in Vancouver, British Columbia, follows a sinusoidal pattern with the highest value of 23.6 °C on day 208, July 26, and the lowest value of 4.2 °C on day 26, January 26.

   a) Use a sine or a cosine function to model the temperatures as a function of time, in days.

   b) From your model, determine the temperature for day 147, May 26.

   c) How many days will have an expected maximum temperature of 21.0 °C or higher?

Extend

22. An investment company invests the money it receives from investors on a collective basis, and each investor shares in the profits and losses. One company has an annual cash flow that has fluctuated in cycles of approximately 40 years since 1920, when it was at a high point. The highs were approximately +20% of the total assets, while the lows were approximately −10% of the total assets.

   a) Model this cash flow as a cosine function of the time, in years, with $t = 0$ representing 1920.

   b) Graph the function from part a).

   c) Determine the cash flow for the company in 2008.

   d) Based on your model, do you feel that this is a company you would invest with? Explain.

23. Golden, British Columbia, is one of the many locations for heliskiing in Western Canada. When skiing the open powder, the skier leaves behind a trail, with two turns creating one cycle of the sinusoidal curve. On one section of the slope, a skier makes a total of 10 turns over a 20-s interval.

   a) If the distance for a turn, to the left or to the right, from the midline is 1.2 m, determine the function that models the path of the skier.

   b) How would the function change if the skier made only eight turns in the same 20-s interval?

Create Connections

C1 **a)** When is it best to use a sine function as a model?

   b) When is it best to use a cosine function as a model?

C2 **a)** Which of the parameters in $y = a \sin b(x - c) + d$ has the greatest influence on the graph of the function? Explain your reasoning.

   b) Which of the parameters in $y = a \cos b(x - c) + d$ has the greatest influence on the graph of the function? Explain your reasoning.

C3 The sinusoidal door by the architectural firm Matharoo Associates is in the home of a diamond merchant in Surat, India. The door measures 5.2 m high and 1.7 m wide. It is constructed from 40 sections of 254-mm-thick Burma teak. Each section is carved so that the door integrates 160 pulleys, 80 ball bearings, a wire rope, and a counterweight hidden within the single pivot. When the door is in an open position, the shape of it may be modelled by a sinusoidal function.

a) Assuming the amplitude is half the width of the door and there is one cycle created within the height of the door, determine a sinusoidal function that could model the shape of the open door.

b) Sketch the graph of your model over one period.

Project Corner **Broadcasting**

- Radio broadcasts, television productions, and cell phone calls are examples of electronic communication.
- A carrier waveform is used in broadcasting the music and voices we hear on the radio. The wave form, which is typically sinusoidal, carries another electrical waveform or message. In the case of AM radio, the sounds (messages) are broadcast through amplitude modulation.

- An NTSC (National Television System Committee) television transmission is comprised of video and sound signals broadcast using carrier waveforms. The video signal is amplitude modulated, while the sound signal is frequency modulated.
- Explain the difference between amplitude modulation and frequency modulation with respect to transformations of functions.
- How are periodic functions involved in satellite radio broadcasting, satellite television broadcasting, or cell phone transmissions?

Chapter 5 Review

5.1 Graphing Sine and Cosine Functions, pages 222–237

1. Sketch the graph of $y = \sin x$ for $-360° \leq x \leq 360°$.

a) What are the x-intercepts?

b) What is the y-intercept?

c) State the domain, range, and period of the function.

d) What is the greatest value of $y = \sin x$?

2. Sketch the graph of $y = \cos x$ for $-360° \leq x \leq 360°$.

a) What are the x-intercepts?

b) What is the y-intercept?

c) State the domain, range, and period of the function.

d) What is the greatest value of $y = \cos x$?

3. Match each function with its correct graph.

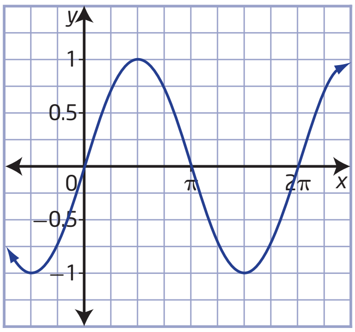
a) $y = \sin x$

b) $y = \sin 2x$

c) $y = -\sin x$

d) $y = \frac{1}{2} \sin x$

A



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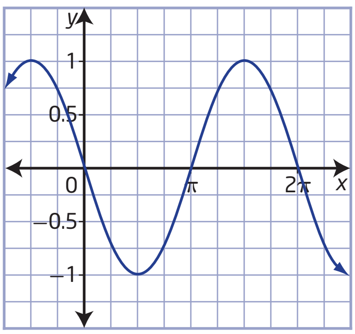

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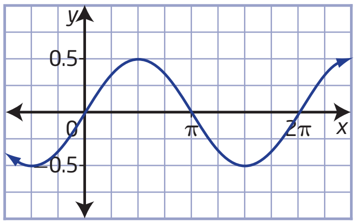
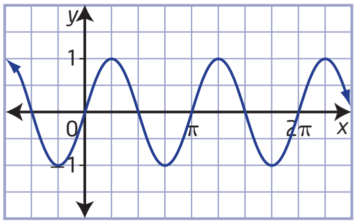


D



4. Without graphing, determine the amplitude and period, in radians and degrees, of each function.

a) $y = -3 \sin 2x$

b) $y = 4 \cos 0.5x$

c) $y = \frac{1}{3} \sin \frac{5}{6}x$

d) $y = -5 \cos \frac{3}{2}x$

5. a) Describe how you could distinguish between the graphs of $y = \sin x$, $y = \sin 2x$, and $y = 2 \sin x$. Graph each function to check your predictions.

b) Describe how you could distinguish between the graphs of $y = \sin x$, $y = -\sin x$, and $y = \sin (-x)$. Graph each function to check your predictions.

c) Describe how you could distinguish between the graphs of $y = \cos x$, $y = -\cos x$, and $y = \cos (-x)$. Graph each function to check your predictions.

6. Write the equation of the cosine function in the form $y = a \cos bx$ with the given characteristics.

a) amplitude 3, period π

b) amplitude 4, period 150°

c) amplitude $\frac{1}{2}$, period 720°

d) amplitude $\frac{3}{4}$, period $\frac{\pi}{6}$

7. Write the equation of the sine function in the form $y = a \sin bx$ with the given characteristics.

a) amplitude 8, period 180°

b) amplitude 0.4, period 60°

c) amplitude $\frac{3}{2}$, period 4π

d) amplitude 2, period $\frac{2\pi}{3}$

5.2 Transformations of Sinusoidal Functions, pages 238–255

8. Determine the amplitude, period, phase shift, and vertical displacement with respect to $y = \sin x$ or $y = \cos x$ for each function. Sketch the graph of each function for two cycles.

a) $y = 2 \cos 3\left(x - \frac{\pi}{2}\right) - 8$

b) $y = \sin \frac{1}{2}\left(x - \frac{\pi}{4}\right) + 3$

c) $y = -4 \cos 2(x - 30°) + 7$

d) $y = \frac{1}{3} \sin \frac{1}{4}(x - 60°) - 1$

9. Sketch graphs of the functions $f(x) = \cos 2\left(x - \frac{\pi}{2}\right)$ and $g(x) = \cos\left(2x - \frac{\pi}{2}\right)$ on the same set of axes for $0 \le x \le 2\pi$.

a) State the period of each function.

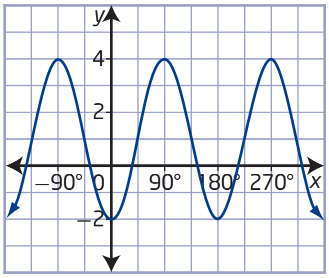
b) State the phase shift for each function.

c) State the phase shift of the function $y = \cos b(x - \pi)$.

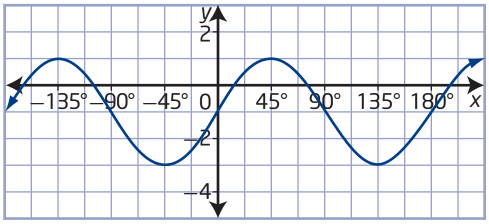
d) State the phase shift of the function $y = \cos (bx - \pi)$.

10. Write the equation for each graph in the form $y = a \sin b(x - c) + d$ and in the form $y = a \cos b(x - c) + d$.

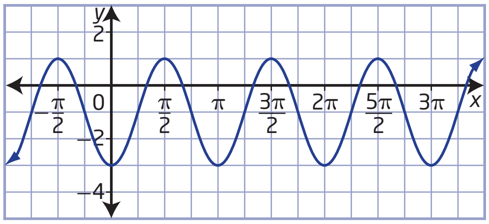
a)



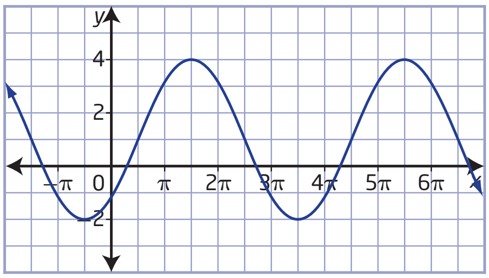
b)



c)



d)



11. a) Write the equation of the sine function with amplitude 4, period π, phase shift $\frac{\pi}{3}$ units to the right, and vertical displacement 5 units down.

b) Write the equation of the cosine function with amplitude 0.5, period 4π, phase shift $\frac{\pi}{6}$ units to the left, and vertical displacement 1 unit up.

c) Write the equation of the sine function with amplitude $\frac{2}{3}$, period 540°, no phase shift, and vertical displacement 5 units down.

12. Graph each function. State the domain, the range, the maximum and minimum values, and the x-intercepts and y-intercept.

a) $y = 2\cos(x - 45°) + 3$

b) $y = 4\sin 2\left(x - \frac{\pi}{3}\right) + 1$

13. Using the language of transformations, describe how to obtain the graph of each function from the graph of $y = \sin x$ or $y = \cos x$.

a) $y = 3\sin 2\left(x - \frac{\pi}{3}\right) + 6$

b) $y = -2\cos \frac{1}{2}\left(x + \frac{\pi}{4}\right) - 3$

c) $y = \frac{3}{4}\cos 2(x - 30°) + 10$

d) $y = -\sin 2(x + 45°) - 8$

14. The sound that the horn of a cruise ship makes as it approaches the dock is different from the sound it makes when it departs. The equation of the sound wave as the ship approaches is $y = 2\sin 2\theta$, while the equation of the sound wave as it departs is $y = 2\sin \frac{1}{2}\theta$.

a) Compare the two sounds by sketching the graphs of the sound waves as the ship approaches and departs for the interval $0 \le \theta \le 2\pi$.

b) How do the two graphs compare to the graph of $y = \sin \theta$?

5.3 The Tangent Function, pages 256–265

15. **a)** Graph $y = \tan \theta$ for $-2\pi \le \theta \le 2\pi$ and for $-360° \le \theta \le 360°$.

b) Determine the following characteristics.

i) domain

ii) range

iii) y-intercept

iv) x-intercepts

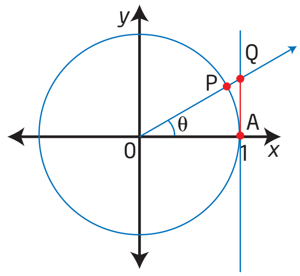
v) equations of the asymptotes

16. A point on the unit circle has coordinates $P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

a) Determine the exact coordinates of point Q.

b) Describe the relationship between $\sin \theta$, $\cos \theta$, and $\tan \theta$.

c) Using the diagram, explain what happens to $\tan \theta$ as θ approaches 90°.



d) What happens to $\tan \theta$ when $\theta = 90°$?

17. **a)** Explain how $\cos \theta$ relates to the asymptotes of the graph of $y = \tan \theta$.

b) Explain how $\sin \theta$ relates to the x-intercepts of the graph of $y = \tan \theta$.

18. Tan θ is sometimes used to measure the lengths of shadows given the angle of elevation of the Sun and the height of a tree. Explain what happens to the shadow of the tree when the Sun is directly overhead. How does this relate to the graph of $y = \tan \theta$?

19. What is a vertical asymptote? How can you tell when a trigonometric function will have a vertical asymptote?

5.4 Equations and Graphs of Trigonometric Functions, pages 266–281

20. Solve each of the following equations graphically.

a) $2\sin x - 1 = 0$, $0 \le x \le 2\pi$

b) $0 = 2\cos(x - 30°) + 5$, $0° \le x \le 360°$

c) $\sin\left(\frac{\pi}{4}(x - 6)\right) = 0.5$, general solution in radians

d) $4\cos(x - 45°) + 7 = 10$, general solution in degrees

21. The Royal British Columbia Museum, home to the First Peoples Exhibit, located in Victoria, British Columbia, was founded in 1886. To preserve the many artifacts, the air-conditioning system in the building operates when the temperature in the building is greater than 22 °C. In the summer, the building's temperature varies with the time of day and is modelled by the function $T = 12 \cos t + 19$, where T represents the temperature in degrees Celsius and t represents the time, in hours.

a) Graph the function.

b) Determine, to the nearest tenth of an hour, the amount of time in one day that the air conditioning will operate.

c) Why is a model for temperature variance important in this situation?

22. The height, h, in metres, above the ground of a rider on a Ferris wheel after t seconds can be modelled by the sine function $h(t) = 12 \sin \frac{\pi}{45}(t - 30) + 15$.

a) Graph the function using graphing technology.

b) Determine the maximum and minimum heights of the rider above the ground.

c) Determine the time required for the Ferris wheel to complete one revolution.

d) Determine the height of the rider above the ground after 45 s.

23. The number of hours of daylight, L, in Lethbridge, Alberta, may be modelled by a sinusoidal function of time, t. The longest day of the year is June 21, with 15.7 h of daylight, and the shortest day is December 21, with 8.3 h of daylight.

a) Determine a sinusoidal function to model this situation.

b) How many hours of daylight are there on April 3?

24. For several hundred years, astronomers have kept track of the number of solar flares, or sunspots, that occur on the surface of the Sun. The number of sunspots counted in a given year varies periodically from a minimum of 10 per year to a maximum of 110 per year. There have been 18 complete cycles between the years 1750 and 1948. Assume that a maximum number of sunspots occurred in the year 1750.

a) How many sunspots would you expect there were in the year 2000?

b) What is the first year after 2000 in which the number of sunspots will be about 35?

c) What is the first year after 2000 in which the number of sunspots will be a maximum?

Chapter 5 Practice Test

Multiple Choice

For #1 to #7, choose the best answer.

1. The range of the function $y = 2 \sin x + 1$ is

A $\{y \mid -1 \le y \le 3, y \in \mathrm{R}\}$

B $\{y \mid -1 \le y \le 1, y \in \mathrm{R}\}$

C $\{y \mid 1 \le y \le 3, y \in \mathrm{R}\}$

D $\{y \mid 0 \le y \le 2, y \in \mathrm{R}\}$

2. What are the phase shift, period, and amplitude, respectively, for the function $f(x) = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 1$?

A $\frac{\pi}{3}, 3, \pi$

B $\pi, \frac{\pi}{3}, 3$

C $3, \frac{\pi}{3}, \pi$

D $\frac{\pi}{3}, \pi, 3$

3. Two functions are given as $f(x) = \sin\left(x - \frac{\pi}{4}\right)$ and $g(x) = \cos (x - a)$. Determine the smallest positive value for a so that the graphs are identical.

A $\frac{\pi}{4}$ **B** $\frac{\pi}{2}$ **C** $\frac{3\pi}{4}$ **D** $\frac{5\pi}{4}$

4. A cosine curve has a maximum point at (3, 14). The nearest minimum point to the right of this maximum point is (8, 2). Which of the following is a possible equation for this curve?

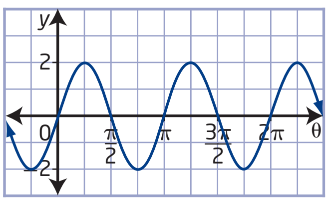
A $y = 6 \cos \frac{2\pi}{5}(x + 3) + 8$

B $y = 6 \cos \frac{2\pi}{5}(x - 3) + 8$

C $y = 6 \cos \frac{\pi}{5}(x + 3) + 8$

D $y = 6 \cos \frac{\pi}{5}(x - 3) + 8$

5. The graph of a sinusoidal function is shown. A possible equation for the function is



A $y = 2 \cos \frac{1}{2}\theta$

B $y = 2 \sin 2\theta$

C $y = 2 \cos 2\theta$

D $y = 2 \sin \frac{1}{2}\theta$

6. Monique makes the following statements about a sine function of the form $y = a \sin b(x - c) + d$:

I The values of a and d affect the range of the function.

II The values of c and d determine the horizontal and vertical translations, respectively.

III The value of b determines the number of cycles within the distance of 2π.

IV The values of a and b are vertical and horizontal stretches.

Monique's correct statements are

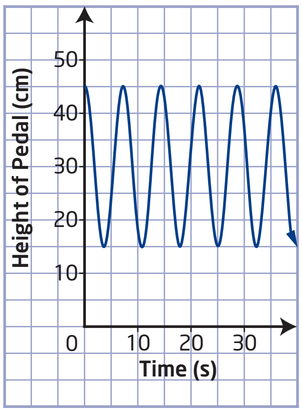
A I, II, III, and IV

B I only

C I, II, and III only

D I, II, and IV only

7. The graph shows how the height of a bicycle pedal changes as the bike is pedalled at a constant speed. How would the graph change if the bicycle were pedalled at a greater constant speed?



A The height of the function would increase.

B The height of the function would decrease.

C The period of the function would decrease.

D The period of the function would increase.

Short Answer

8. What is the horizontal distance between two consecutive zeros of the function $f(x) = \sin 2x$?

9. For the function $y = \tan \theta$, state the asymptotes, domain, range, and period.

10. What do the functions $f(x) = -4 \sin x$ and $g(x) = -4 \cos \frac{1}{2}x$ have in common?

11. An airplane's electrical generator produces a time-varying output voltage described by the equation $V(t) = 120 \sin 2513t$, where t is the time, in seconds, and V is in volts. What are the amplitude and period of this function?

12. Suppose the depth, d, in metres, of the tide in a certain harbour can be modelled by $d(t) = -3 \cos \frac{\pi}{6}t + 5$, where t is the time, in hours. Consider a day in which $t = 0$ represents the time 00:00. Determine the time for the high and low tides and the depths of each.

13. Solve each of the following equations graphically.

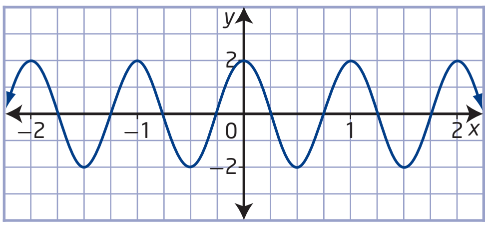
a) $\sin\left(\frac{\pi}{3}(x - 1)\right) = 0.5$, general solution in radians

b) $4 \cos (15(x + 30°)) + 1 = -2$, general solution in degrees

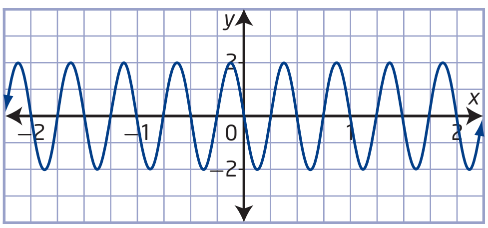
Extended Response

14. Compare and contrast the two graphs of sinusoidal functions.

I



II



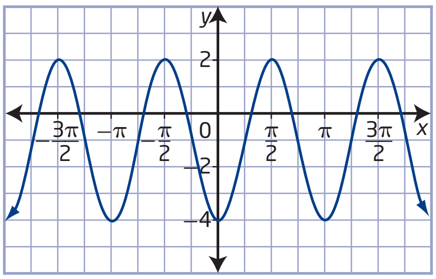
15. Suppose a mass suspended on a spring is bouncing up and down. The mass's distance from the floor when it is at rest is 1 m. The maximum displacement is 10 cm as it bounces. It takes 2 s to complete one bounce or cycle. Suppose the mass is at rest at $t = 0$ and that the spring bounces up first.

a) Write a function to model the displacement as a function of time.

b) Graph the function to determine the approximate times when the mass is 1.05 m above the floor in the first cycle.

c) Verify your solutions to part b) algebraically.

16. The graph of a sinusoidal function is shown.



a) Determine a function for the graph in the form $y = a \sin b(x - c) + d$.

b) Determine a function for the graph in the form $y = a \cos b(x - c) + d$.

17. A student is investigating the effects of changing the values of the parameters a, b, c, and d in the function $y = a \sin b(x - c) + d$. The student graphs the following functions:

A $f(x) = \sin x$

B $g(x) = 2 \sin x$

C $h(x) = \sin 2x$

D $k(x) = \sin (2x + 2)$

E $m(x) = \sin 2x + 2$

a) Which graphs have the same x-intercepts?

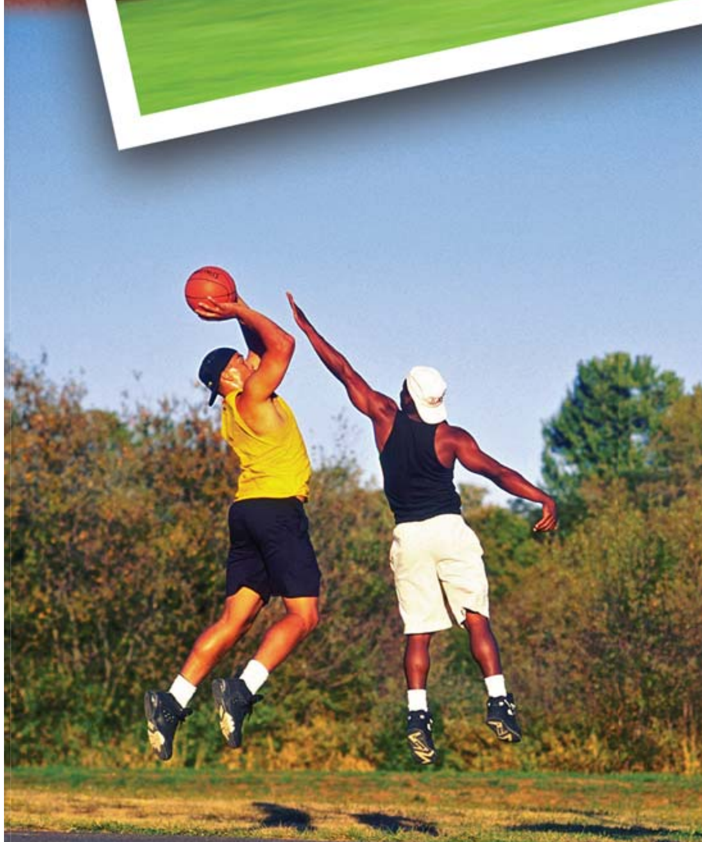
b) Which graphs have the same period?

c) Which graph has a different amplitude than the others?

CHAPTER
6

Trigonometric Identities

Trigonometric functions are used to model behaviour in the physical world. You can model projectile motion, such as the path of a thrown javelin or a lobbed tennis ball with trigonometry. Sometimes equivalent expressions for trigonometric functions can be substituted to allow scientists to analyse data or solve a problem more efficiently. In this chapter, you will explore equivalent trigonometric expressions.



Did You Know?

Elizabeth Gleadle, of Vancouver, British Columbia, holds the Canadian women's javelin record, with a distance of 58.21 m thrown in July 2009.

Key Terms

trigonometric identity

Career Link

An athletic therapist works with athletes to prevent, assess, and rehabilitate sports-related injuries, and facilitate a return to competitive sport after injury. Athletic therapists can begin their careers by obtaining a Bachelor of Kinesiology from an institution such as the University of Calgary. This degree can provide entrance to medical schools and eventually sports medicine as a specialty.

Web Link

To learn more about kinesiology and a career as an athletic therapist, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

6.1

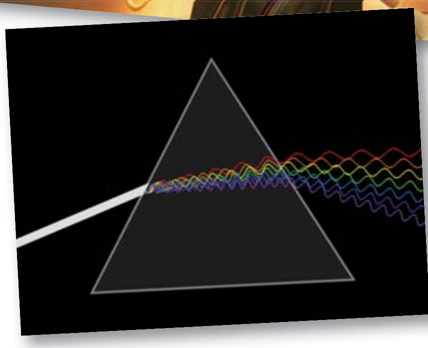
Reciprocal, Quotient, and Pythagorean Identities

Engineer using an electronic spin resonance spectroscope

Focus on...

- verifying a trigonometric identity numerically and graphically using technology
- exploring reciprocal, quotient, and Pythagorean identities
- determining non-permissible values of trigonometric identities
- explaining the difference between a trigonometric identity and a trigonometric equation

Digital music players store large sound files by using trigonometry to compress (store) and then decompress (play) the file when needed. A large sound file can be stored in a much smaller space using this technique. Electronics engineers have learned how to use the periodic nature of music to compress the audio file into a smaller space.

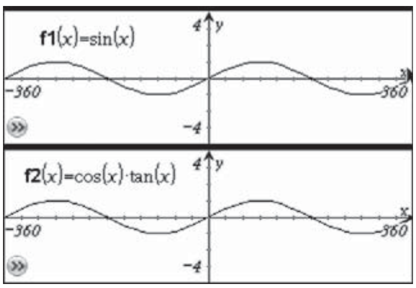


Investigate Comparing Two Trigonometric Expressions

Materials

- graphing technology

1. Graph the curves $y = \sin x$ and $y = \cos x \tan x$ over the domain $-360° \leq x < 360°$. Graph the curves on separate grids using the same range and scale. What do you notice?



2. Make and analyse a table of values for these functions in multiples of 30° over the domain $-360° \leq x < 360°$. Describe your findings.

3. Use your knowledge of $\tan x$ to simplify the expression $\cos x \tan x$.

Reflect and Respond

4. **a)** Are the curves $y = \sin x$ and $y = \cos x \tan x$ identical? Explain your reasoning.

   b) Why was it important to look at the graphs *and* at the table of values?

5. What are the non-permissible values of x in the equation $\sin x = \cos x \tan x$? Explain.

6. Are there any permissible values for x outside the domain in step 2 for which the expressions $\sin x$ and $\cos x \tan x$ are not equal? Share your response with a classmate.

Link the Ideas

The equation $\sin x = \cos x \tan x$ that you explored in the investigation is an example of a **trigonometric identity**. Both sides of the equation have the same value for all permissible values of x. In other words, when the expressions on either side of the equal sign are evaluated for any permissible value, the resulting values are equal. Trigonometric identities can be verified both numerically and graphically.

trigonometric identity

- a trigonometric equation that is true for all permissible values of the variable in the expressions on both sides of the equation

You are familiar with two groups of identities from your earlier work with trigonometry: the reciprocal identities and the quotient identity.

Reciprocal Identities

$\csc x = \dfrac{1}{\sin x}$ $\qquad \sec x = \dfrac{1}{\cos x}$ $\qquad \cot x = \dfrac{1}{\tan x}$

Quotient Identities

$\tan x = \dfrac{\sin x}{\cos x}$ $\qquad \cot x = \dfrac{\cos x}{\sin x}$

Example 1

Verify a Potential Identity Numerically and Graphically

a) Determine the non-permissible values, in degrees, for the equation $\sec \theta = \dfrac{\tan \theta}{\sin \theta}$.

b) Numerically verify that $\theta = 60°$ and $\theta = \dfrac{\pi}{4}$ are solutions of the equation.

c) Use technology to graphically decide whether the equation could be an identity over the domain $-360° < \theta \leq 360°$.

Solution

a) To determine the non-permissible values, assess each trigonometric function in the equation individually and examine expressions that may have non-permissible values. Visualize the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ to help you determine the non-permissible values.

First consider the left side, $\sec \theta$:

$\sec \theta = \dfrac{1}{\cos \theta}$, and $\cos \theta = 0$ when $\theta = 90°, 270°, \ldots$.

So, the non-permissible values for $\sec \theta$ are $\theta \neq 90° + 180°n$, where $n \in I$.

Now consider the right side, $\dfrac{\tan \theta}{\sin \theta}$:

$\tan \theta$ is not defined when $\theta = 90°, 270°, \ldots$.

Why must these values be excluded?

So, the non-permissible values for $\tan \theta$ are $\theta \neq 90° + 180°n$, where $n \in I$.

How do these non-permissible values compare to the ones found for the left side?

Also, the expression $\dfrac{\tan \theta}{\sin \theta}$ is undefined when $\sin \theta = 0$.

$\sin \theta = 0$ when $\theta = 0°, 180°, \ldots$.

So, further non-permissible values for $\dfrac{\tan \theta}{\sin \theta}$ are $\theta \neq 180°n$, where $n \in I$.

Are these non-permissible values included in the ones already found?

The three sets of non-permissible values for the equation $\sec \theta = \dfrac{\tan \theta}{\sin \theta}$ can be expressed as a single restriction, $\theta \neq 90°n$, where $n \in I$.

b) Substitute $\theta = 60°$.

$$\begin{aligned}\text{Left Side} &= \sec \theta \\ &= \sec 60° \\ &= \frac{1}{\cos 60°} \\ &= \frac{1}{0.5} \\ &= 2\end{aligned}$$

$$\begin{aligned}\text{Right Side} &= \frac{\tan \theta}{\sin \theta} \\ &= \frac{\tan 60°}{\sin 60°} \\ &= \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} \\ &= 2\end{aligned}$$

Why does substituting 60° in both sides of the equation not prove that the identity is true?

Left Side = Right Side

The equation $\sec \theta = \dfrac{\tan \theta}{\sin \theta}$ is true for $\theta = 60°$.

Substitute $\theta = \dfrac{\pi}{4}$.

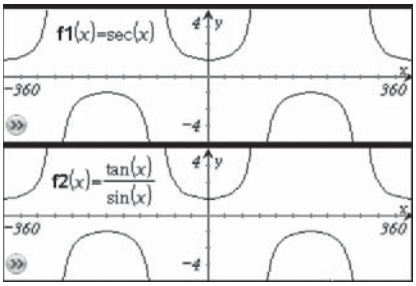
$$\begin{aligned}\text{Left Side} &= \sec \theta \\ &= \sec \frac{\pi}{4} \\ &= \frac{1}{\cos \frac{\pi}{4}} \\ &= \frac{1}{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\text{Right Side} &= \frac{\tan \theta}{\sin \theta} \\ &= \frac{\tan \frac{\pi}{4}}{\sin \frac{\pi}{4}} \\ &= \frac{1}{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2}\end{aligned}$$

Left Side = Right Side

The equation $\sec \theta = \dfrac{\tan \theta}{\sin \theta}$ is true for $\theta = \dfrac{\pi}{4}$.

c) Use technology, with domain $-360° < x \leq 360°$, to graph $y = \sec \theta$ and $y = \frac{\tan \theta}{\sin \theta}$. The graphs look identical, so $\sec \theta = \frac{\tan \theta}{\sin \theta}$ could be an identity.



How do these graphs show that there are non-permissible values for this identity?

Does graphing the related functions on each side of the equation prove that the identity is true? Explain.

Your Turn

a) Determine the non-permissible values, in degrees, for the equation $\cot x = \frac{\cos x}{\sin x}$.

b) Verify that $x = 45°$ and $x = \frac{\pi}{6}$ are solutions to the equation.

c) Use technology to graphically decide whether the equation could be an identity over the domain $-360° < x \leq 360°$.

Example 2

Use Identities to Simplify Expressions

a) Determine the non-permissible values, in radians, of the variable in the expression $\frac{\cot x}{\csc x \cos x}$.

b) Simplify the expression.

Solution

a) The trigonometric functions cot x and csc x both have non-permissible values in their domains.

For $\cot x$, $x \neq \pi n$, where $n \in \mathrm{I}$.
For $\csc x$, $x \neq \pi n$, where $n \in \mathrm{I}$.

Why are these the non-permissible values for both reciprocal functions?

Also, the denominator of $\frac{\cot x}{\csc x \cos x}$ cannot equal zero. In other words, $\csc x \cos x \neq 0$.

There are no values of x that result in $\csc x = 0$.

However, for $\cos x$, $x \neq \frac{\pi}{2} + \pi n$, where $n \in \mathrm{I}$.

Combined, the non-permissible values for $\frac{\cot x}{\csc x \cos x}$ are $x \neq \frac{\pi}{2}n$, where $n \in \mathrm{I}$.

Why can you write this single general restriction?

b) To simplify the expression, use reciprocal and quotient identities to write trigonometric functions in terms of cosine and sine.

$$\frac{\cot x}{\csc x \cos x} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}\cos x}$$

$$= \frac{\frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x}} \quad \text{Simplify the fraction.}$$

$$= 1$$

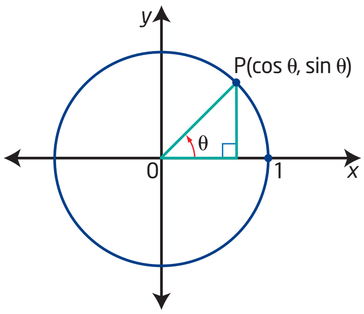
Your Turn

a) Determine the non-permissible values, in radians, of the variable in the expression $\frac{\sec x}{\tan x}$.

b) Simplify the expression.

Pythagorean Identity

Recall that point P on the terminal arm of an angle θ in standard position has coordinates (cos θ, sin θ). Consider a right triangle with a hypotenuse of 1 and legs of cos θ and sin θ.



The hypotenuse is 1 because it is the radius of the unit circle. Apply the Pythagorean theorem in the right triangle to establish the Pythagorean identity:

$$x^2 + y^2 = 1^2$$
$$\cos^2 \theta + \sin^2 \theta = 1$$

Example 3

Use the Pythagorean Identity

a) Verify that the equation $\cot^2 x + 1 = \csc^2 x$ is true when $x = \frac{\pi}{6}$.

b) Use quotient identities to express the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ as the equivalent identity $\cot^2 x + 1 = \csc^2 x$.

Solution

a) Substitute $x = \frac{\pi}{6}$.

$$\begin{aligned}\text{Left Side} &= \cot^2 x + 1\\ &= \cot^2 \frac{\pi}{6} + 1\\ &= \frac{1}{\tan^2 \frac{\pi}{6}} + 1\\ &= \frac{1}{\frac{1}{(\sqrt{3})^2}} + 1\\ &= (\sqrt{3})^2 + 1\\ &= 4\end{aligned}
\qquad
\begin{aligned}\text{Right Side} &= \csc^2 x\\ &= \csc^2 \frac{\pi}{6}\\ &= \frac{1}{\sin^2 \frac{\pi}{6}}\\ &= \frac{1}{\left(\frac{1}{2}\right)^2}\\ &= 2^2\\ &= 4\end{aligned}$$

$$\text{Left Side} = \text{Right Side}$$

The equation $\cot^2 x + 1 = \csc^2 x$ is true when $x = \frac{\pi}{6}$.

b) $\cos^2 x + \sin^2 x = 1$

Since this identity is true for all permissible values of x, you can multiply both sides by $\frac{1}{\sin^2 x}$, $x \neq \pi n$, where $n \in \mathrm{I}$.

$$\begin{aligned}\left(\frac{1}{\sin^2 x}\right)\cos^2 x + \left(\frac{1}{\sin^2 x}\right)\sin^2 x &= \left(\frac{1}{\sin^2 x}\right)1\\ \frac{\cos^2 x}{\sin^2 x} + 1 &= \frac{1}{\sin^2 x}\\ \cot^2 x + 1 &= \csc^2 x\end{aligned}$$

Why multiply both sides by $\frac{1}{\sin^2 x}$? How else could you simplify this equation?

Your Turn

a) Verify the equation $1 + \tan^2 x = \sec^2 x$ numerically for $x = \frac{3\pi}{4}$.

b) Express the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ as the equivalent identity $1 + \tan^2 x = \sec^2 x$.

The three forms of the Pythagorean identity are

$\cos^2 \theta + \sin^2 \theta = 1$ $\qquad \cot^2 \theta + 1 = \csc^2 \theta$ $\qquad 1 + \tan^2 \theta = \sec^2 \theta$

Key Ideas

- A trigonometric identity is an equation involving trigonometric functions that is true for all permissible values of the variable.
- You can verify trigonometric identities
  - numerically by substituting specific values for the variable
  - graphically, using technology
- Verifying that two sides of an equation are equal for given values, or that they appear equal when graphed, is not sufficient to conclude that the equation is an identity.
- You can use trigonometric identities to simplify more complicated trigonometric expressions.
- The reciprocal identities are

  $\csc x = \frac{1}{\sin x}$ $\quad \sec x = \frac{1}{\cos x}$ $\quad \cot x = \frac{1}{\tan x}$
- The quotient identities are

  $\tan x = \frac{\sin x}{\cos x}$ $\quad \cot x = \frac{\cos x}{\sin x}$
- The Pythagorean identities are

  $\cos^2 x + \sin^2 x = 1$ $\quad 1 + \tan^2 x = \sec^2 x$ $\quad \cot^2 x + 1 = \csc^2 x$

Check Your Understanding

Practise

1. Determine the non-permissible values of x, in radians, for each expression.

a) $\frac{\cos x}{\sin x}$ **b)** $\frac{\sin x}{\tan x}$

c) $\frac{\cot x}{1 - \sin x}$ **d)** $\frac{\tan x}{\cos x + 1}$

2. Why do some identities have non-permissible values?

3. Simplify each expression to one of the three primary trigonometric functions, $\sin x$, $\cos x$ or $\tan x$. For part a), verify graphically, using technology, that the given expression is equivalent to its simplified form.

a) $\sec x \sin x$

b) $\sec x \cot x \sin^2 x$

c) $\frac{\cos x}{\cot x}$

4. Simplify, and then rewrite each expression as one of the three reciprocal trigonometric functions, $\csc x$, $\sec x$, or $\cot x$.

a) $\left(\frac{\cos x}{\tan x}\right)\left(\frac{\tan x}{\sin x}\right)$

b) $\csc x \cot x \sec x \sin x$

c) $\frac{\cos x}{1 - \sin^2 x}$

5. a) Verify that the equation $\frac{\sec x}{\tan x + \cot x} = \sin x$ is true for $x = 30°$ and for $x = \frac{\pi}{4}$.

b) What are the non-permissible values of the equation in the domain $0° \leq x < 360°$?

6. Consider the equation
$\dfrac{\sin x \cos x}{1 + \cos x} = \dfrac{1 - \cos x}{\tan x}$.

a) What are the non-permissible values, in radians, for this equation?

b) Graph the two sides of the equation using technology, over the domain $0 \leq x < 2\pi$. Could it be an identity?

c) Verify that the equation is true when $x = \frac{\pi}{4}$. Use exact values for each expression in the equation.

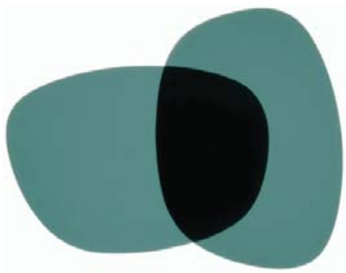
Apply

7. When a polarizing lens is rotated through an angle θ over a second lens, the amount of light passing through both lenses decreases by $1 - \sin^2 \theta$.

a) Determine an equivalent expression for this decrease using only cosine.

b) What fraction of light is lost when $\theta = \frac{\pi}{6}$?

c) What percent of light is lost when $\theta = 60°$?



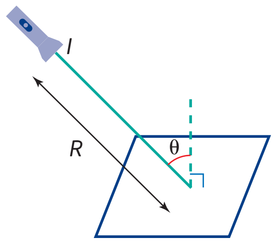
8. Compare $y = \sin x$ and $y = \sqrt{1 - \cos^2 x}$ by completing the following.

a) Verify that $\sin x = \sqrt{1 - \cos^2 x}$ for $x = \frac{\pi}{3}$, $x = \frac{5\pi}{6}$, and $x = \pi$.

b) Graph $y = \sin x$ and $y = \sqrt{1 - \cos^2 x}$ in the same window.

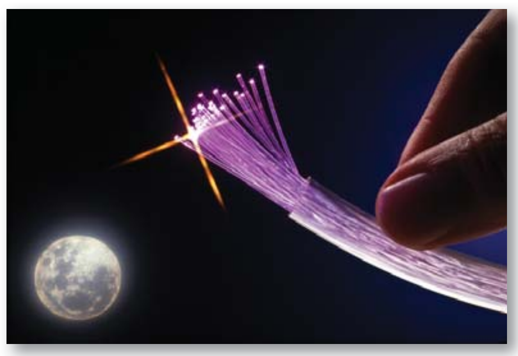
c) Determine whether $\sin x = \sqrt{1 - \cos^2 x}$ is an identity. Explain your answer.

9. Illuminance (E) is a measure of the amount of light coming from a light source and falling onto a surface. If the light is projected onto the surface at an angle θ, measured from the perpendicular, then a formula relating these values is $\sec \theta = \frac{I}{ER^2}$, where I is a measure of the luminous intensity and R is the distance between the light source and the surface.



a) Rewrite the formula so that E is isolated and written in terms of $\cos \theta$.

b) Show that $E = \dfrac{I \cot \theta}{R^2 \csc \theta}$ is equivalent to your equation from part a).



Fibre optic cable

10. Simplify $\dfrac{\csc x}{\tan x + \cot x}$ to one of the three primary trigonometric ratios. What are the non-permissible values of the original expression in the domain $0 \leq x < 2\pi$?

11. a) Determine graphically, using technology, whether the expression $\frac{\csc^2 x - \cot^2 x}{\cos x}$ appears to be equivalent to $\csc x$ or $\sec x$.

b) What are the non-permissible values, in radians, for the identity from part a)?

c) Express $\frac{\csc^2 x - \cot^2 x}{\cos x}$ as the single reciprocal trigonometric ratio that you identified in part a).

12. a) Substitute $x = \frac{\pi}{4}$ into the equation $\frac{\cot x}{\sec x} + \sin x = \csc x$ to determine whether it could be an identity. Use exact values.

b) Algebraically confirm that the expression on the left side simplifies to $\csc x$.

13. Stan, Lina, and Giselle are working together to try to determine whether the equation $\sin x + \cos x = \tan x + 1$ is an identity.

a) Stan substitutes $x = 0$ into each side of the equation. What is the result?

b) Lina substitutes $x = \frac{\pi}{2}$ into each side of the equation. What does she observe?

c) Stan points out that Lina's choice is not permissible for this equation. Explain why.

d) Giselle substitutes $x = \frac{\pi}{4}$ into each side of the equation. What does she find?

e) Do the three students have enough information to conclude whether or not the given equation is an identity? Explain.

14. Simplify $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$.

Extend

15. Given $\csc^2 x + \sin^2 x = 7.89$, find the value of $\frac{1}{\csc^2 x} + \frac{1}{\sin^2 x}$.

16. Show algebraically that
$$\frac{1}{1 + \sin \theta} + \frac{1}{1 - \sin \theta} = 2 \sec^2 \theta$$
is an identity.

17. Determine an expression for m that makes $\frac{2 - \cos^2 x}{\sin x} = m + \sin x$ an identity.

Create Connections

C1 Explain how a student who does not know the $\cot^2 x + 1 = \csc^2 x$ form of the Pythagorean identity could simplify an expression that contained the expression $\cot^2 x + 1$ using the fact that $1 = \frac{\sin^2 x}{\sin^2 x}$.

C2 For some trigonometric expressions, multiplying by a conjugate helps to simplify the expression. Simplify $\frac{\sin \theta}{1 + \cos \theta}$ by multiplying the numerator and the denominator by the conjugate of the denominator, $1 - \cos \theta$. Describe how this process helps to simplify the expression.

C3 MINI LAB Explore the effect of different domains on apparent identities.

Materials
- graphing calculator

Step 1 Graph the two functions $y = \tan x$ and $y = \left|\frac{\sin x}{\cos x}\right|$ on the same grid, using a domain of $0 \le x < \frac{\pi}{2}$. Is there graphical evidence that $\tan x = \left|\frac{\sin x}{\cos x}\right|$ is an identity? Explain.

Step 2 Graph the two functions $y = \tan x$ and $y = \left|\frac{\sin x}{\cos x}\right|$ again, using the expanded domain $-2\pi < x \le 2\pi$. Is the equation $\tan x = \left|\frac{\sin x}{\cos x}\right|$ an identity? Explain.

Step 3 Find and record a different trigonometric equation that is true over a restricted domain but is not an identity when all permissible values are checked. Compare your answer with that of a classmate.

Step 4 How does this activity show the weakness of using graphical and numerical methods for verifying potential identities?

6.2

Sum, Difference, and Double-Angle Identities

Focus on...

- applying sum, difference, and double-angle identities to verify the equivalence of trigonometric expressions
- verifying a trigonometric identity numerically and graphically using technology


Paris gold box

In addition to holograms and security threads, paper money often includes special Guilloché patterns in the design to prevent counterfeiting. The sum and product of nested sinusoidal functions are used to form the blueprint of some of these patterns. Guilloché patterns have been created since the sixteenth century, but their origin is uncertain. They can be found carved in wooden door frames and etched on the metallic surfaces of objects such as vases.

Web Link

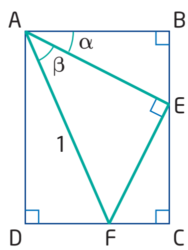
To learn more about Guilloché patterns, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

Investigate Expressions for sin $(\alpha + \beta)$ and cos $(\alpha + \beta)$

Materials

- ruler
- protractor

1. a) Draw a large rectangle and label its vertices A, B, C, and D, where BC < 2AB. Mark a point E on BC. Join AE and use a protractor to draw EF perpendicular to AE. Label all right angles on your diagram. Label ∠BAE as α and ∠EAF as β.



b) Measure the angles α and β. Use the angle sum of a triangle to determine the measures of all the remaining acute angles in your diagram. Record their measures on the diagram.

2. a) Explain how you know that $\angle CEF = \alpha$.

b) Determine an expression for each of the other acute angles in the diagram in terms of α and β. Label each angle on your diagram.

3. Suppose the hypotenuse AF of the inscribed right triangle has a length of 1 unit. Explain why the length of AE can be represented as $\cos \beta$. Label AE as $\cos \beta$.
4. Determine expressions for line segments AB, BE, EF, CE, CF, AD, and DF in terms of $\sin \alpha$, $\cos \alpha$, $\sin \beta$, and $\cos \beta$. Label each side length on your diagram using these sines and cosines. Note that AD equals the sum of segments BE and EC, and DF equals AB minus CF.
5. Which angle in the diagram is equivalent to $\alpha + \beta$? Determine possible identities for $\sin (\alpha + \beta)$ and $\cos (\alpha + \beta)$ from $\triangle$ADF using the sum or difference of lengths. Compare your results with those of a classmate.

Reflect and Respond

6. a) Verify your possible identities numerically using the measures of α and β from step 1. Compare your results with those of a classmate.
   b) Does each identity apply to angles that are obtuse? Are there any restrictions on the domain? Describe your findings.
7. Consider the special case where $\alpha = \beta$. Write simplified equivalent expressions for $\sin 2\alpha$ and $\cos 2\alpha$.

Link the Ideas

In the investigation, you discovered the angle sum identities for sine and cosine. These identities can be used to determine the angle sum identity for tangent.

The sum identities are

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

The angle sum identities for sine, cosine, and tangent can be used to determine angle difference identities for sine, cosine, and tangent.

For sine,

$$\begin{aligned}\sin (A - B) &= \sin (A + (-B))\\ &= \sin A \cos (-B) + \cos A \sin (-B)\\ &= \sin A \cos B + \cos A (-\sin B) && \text{Why is } \cos(-B) = \cos B?\\ &= \sin A \cos B - \cos A \sin B && \text{Why is } \sin(-B) = -\sin B?\end{aligned}$$

Web Link

To see a derivation of the difference cos (A − B), go to www.mcgrawhill.ca/school/learningcentres and follow the links.

The three angle difference identities are

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

A special case occurs in the angle sum identities when A = B. Substituting B = A results in the double-angle identities.

For example,
$$\begin{aligned}\sin 2A &= \sin (A + A)\\ &= \sin A \cos A + \cos A \sin A\\ &= 2 \sin A \cos A\end{aligned}$$

Similarly, it can be shown that
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

The double-angle identities are
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Example 1

Simplify Expressions Using Sum, Difference, and Double-Angle Identities

Write each expression as a single trigonometric function.

a) $\sin 48° \cos 17° - \cos 48° \sin 17°$

b) $\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$

Solution

a) The expression $\sin 48° \cos 17° - \cos 48° \sin 17°$ has the same form as the right side of the difference identity for sine, $\sin (A - B) = \sin A \cos B - \cos A \sin B$.

Thus,
$$\begin{aligned}\sin 48° \cos 17° - \cos 48° \sin 17° &= \sin (48° - 17°)\\ &= \sin 31°\end{aligned}$$

b) The expression $\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$ has the same form as the right side of the double-angle identity for cosine, $\cos 2A = \cos^2 A - \sin^2 A$.

Therefore,
$$\begin{aligned}\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3} &= \cos \left(2\left(\frac{\pi}{3}\right)\right)\\ &= \cos \frac{2\pi}{3}\end{aligned}$$

How could you use technology to verify these solutions?

Your Turn

Write each expression as a single trigonometric function.

a) $\cos 88° \cos 35° + \sin 88° \sin 35°$

b) $2 \sin \frac{\pi}{12} \cos \frac{\pi}{12}$

Example 2

Determine Alternative Forms of the Double-Angle Identity for Cosine

Determine an identity for cos 2A that contains only the cosine ratio.

Solution

An identity for cos 2A is $\cos 2A = \cos^2 A - \sin^2 A$.

Write an equivalent expression for the term containing sin A.

Use the Pythagorean identity, $\cos^2 A + \sin^2 A = 1$.

Substitute $\sin^2 A = 1 - \cos^2 A$ to obtain another form of the double-angle identity for cosine.

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \cos^2 A - (1 - \cos^2 A)\\ &= \cos^2 A - 1 + \cos^2 A\\ &= 2\cos^2 A - 1\end{aligned}$$

Your Turn

Determine an identity for cos 2A that contains only the sine ratio.

Example 3

Simplify Expressions Using Identities

Consider the expression $\dfrac{1 - \cos 2x}{\sin 2x}$.

a) What are the permissible values for the expression?

b) Simplify the expression to one of the three primary trigonometric functions.

c) Verify your answer from part b), in the interval $[0, 2\pi)$, using technology.

Solution

a) Identify any non-permissible values. The expression is undefined when $\sin 2x = 0$.

Method 1: Simplify the Double Angle

Use the double-angle identity for sine to simplify $\sin 2x$ first.

$\sin 2x = 2 \sin x \cos x$
$2 \sin x \cos x \neq 0$
So, $\sin x \neq 0$ and $\cos x \neq 0$.
$\sin x = 0$ when $x = \pi n$, where $n \in I$.
$\cos x = 0$ when $x = \dfrac{\pi}{2} + \pi n$, where $n \in I$.
When these two sets of non-permissible values are combined, the permissible values for the expression are all real numbers except $x \neq \dfrac{\pi n}{2}$, where $n \in I$.

Method 2: Horizontal Transformation of sin x

First determine when $\sin x = 0$. Then, stretch the domain horizontally by a factor of $\frac{1}{2}$.

$\sin x = 0$ when $x = \pi n$, where $n \in \mathrm{I}$.

Therefore, $\sin 2x = 0$ when $x = \frac{\pi n}{2}$, where $n \in \mathrm{I}$.

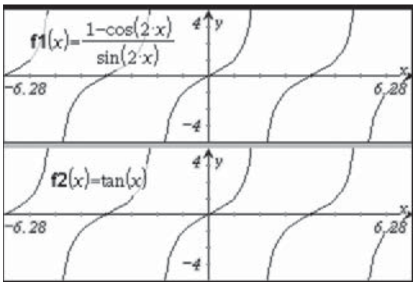
The permissible values of the expression $\frac{1 - \cos 2x}{\sin 2x}$ are all real numbers except $x \neq \frac{\pi n}{2}$, where $n \in \mathrm{I}$.

b)
$$\begin{aligned} \frac{1 - \cos 2x}{\sin 2x} &= \frac{1 - (1 - 2\sin^2 x)}{2 \sin x \cos x} \\ &= \frac{2 \sin^2 x}{2 \sin x \cos x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}$$

Replace sin $2x$ in the denominator. Replace cos $2x$ with the form of the identity from Example 2 that will simplify most fully.

The expression $\frac{1 - \cos 2x}{\sin 2x}$ is equivalent to $\tan x$.

c) Use technology, with domain $0 \leq x < 2\pi$, to graph $y = \frac{1 - \cos 2x}{\sin 2x}$ and $y = \tan x$. The graphs look identical, which verifies, but does not prove, the answer in part b).



Your Turn

Consider the expression $\frac{\sin 2x}{\cos 2x + 1}$.

a) What are the permissible values for the expression?

b) Simplify the expression to one of the three primary trigonometric functions.

c) Verify your answer from part b), in the interval $[0, 2\pi)$, using technology.

Example 4

Determine Exact Trigonometric Values for Angles

Determine the exact value for each expression.

a) $\sin \frac{\pi}{12}$

b) $\tan 105°$

Solution

a) Use the difference identity for sine with two special angles.

For example, because $\frac{\pi}{12} = \frac{3\pi}{12} - \frac{2\pi}{12}$, use $\frac{\pi}{4} - \frac{\pi}{6}$.

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin \left(\frac{\pi}{4} - \frac{\pi}{6}\right)\\ &= \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6}\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

The special angles $\frac{\pi}{3}$ and $\frac{\pi}{4}$ could also be used.

Use $\sin (A - B) = \sin A \cos B - \cos A \sin B$.

How could you verify this answer with a calculator?

b) **Method 1: Use the Difference Identity for Tangent**

Rewrite tan 105° as a difference of special angles.

$\tan 105° = \tan (135° - 30°)$

Are there other ways of writing 105° as the sum or difference of two special angles?

Use the tangent difference identity, $\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

$$\begin{aligned}\tan (135° - 30°) &= \frac{\tan 135° - \tan 30°}{1 + \tan 135° \tan 30°}\\ &= \frac{-1 - \frac{1}{\sqrt{3}}}{1 + (-1)\left(\frac{1}{\sqrt{3}}\right)}\\ &= \frac{-1 - \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}\\ &= \left(\frac{-1 - \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}\right)\left(\frac{-\sqrt{3}}{-\sqrt{3}}\right)\\ &= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}\end{aligned}$$

Simplify.

Multiply numerator and denominator by $-\sqrt{3}$.

How could you rationalize the denominator?

Method 2: Use a Quotient Identity with Sine and Cosine

$$\begin{aligned}
\tan 105° &= \frac{\sin 105°}{\cos 105°} \\
&= \frac{\sin (60° + 45°)}{\cos (60° + 45°)} \\
&= \frac{\sin 60° \cos 45° + \cos 60° \sin 45°}{\cos 60° \cos 45° - \sin 60° \sin 45°} \\
&= \frac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)} \\
&= \frac{\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}}{\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}} \\
&= \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)\left(\frac{4}{\sqrt{2} - \sqrt{6}}\right) \\
&= \frac{\sqrt{6} + \sqrt{2}}{\sqrt{2} - \sqrt{6}}
\end{aligned}$$

Use sum identities with special angles. Could you use a difference of angles identity here?

How could you verify that this is the same answer as in Method 1?

Your Turn

Use a sum or difference identity to find the exact values of

a) $\cos 165°$

b) $\tan \frac{11\pi}{12}$

Key Ideas

- You can use the sum and difference identities to simplify expressions and to determine exact trigonometric values for some angles.

Sum Identities

$\sin (A + B) = \sin A \cos B + \cos A \sin B$

$\cos (A + B) = \cos A \cos B - \sin A \sin B$

$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

Difference Identities

$\sin (A - B) = \sin A \cos B - \cos A \sin B$

$\cos (A - B) = \cos A \cos B + \sin A \sin B$

$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

- The double-angle identities are special cases of the sum identities when the two angles are equal. The double-angle identity for cosine can be expressed in three forms using the Pythagorean identity, $\cos^2 A + \sin^2 A = 1$.

Double-Angle Identities

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$
$\cos 2A = 2 \cos^2 A - 1$
$\cos 2A = 1 - 2 \sin^2 A$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

Check Your Understanding

Practise

1. Write each expression as a single trigonometric function.

a) $\cos 43° \cos 27° - \sin 43° \sin 27°$

b) $\sin 15° \cos 20° + \cos 15° \sin 20°$

c) $\cos^2 19° - \sin^2 19°$

d) $\sin \frac{3\pi}{2} \cos \frac{5\pi}{4} - \cos \frac{3\pi}{2} \sin \frac{5\pi}{4}$

e) $8 \sin \frac{\pi}{3} \cos \frac{\pi}{3}$

2. Simplify and then give an exact value for each expression.

a) $\cos 40° \cos 20° - \sin 40° \sin 20°$

b) $\sin 20° \cos 25° + \cos 20° \sin 25°$

c) $\cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$

d) $\cos \frac{\pi}{2} \cos \frac{\pi}{3} - \sin \frac{\pi}{2} \sin \frac{\pi}{3}$

3. Using only one substitution, which form of the double-angle identity for cosine will simplify the expression $1 - \cos 2x$ to one term? Show how this happens.

4. Write each expression as a single trigonometric function.

a) $2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}$

b) $(6 \cos^2 24° - 6 \sin^2 24°) \tan 48°$

c) $\frac{2 \tan 76°}{1 - \tan^2 76°}$

d) $2 \cos^2 \frac{\pi}{6} - 1$

e) $1 - 2 \cos^2 \frac{\pi}{12}$

5. Simplify each expression to a single primary trigonometric function.

a) $\frac{\sin 2\theta}{2 \cos \theta}$

b) $\cos 2x \cos x + \sin 2x \sin x$

c) $\frac{\cos 2\theta + 1}{2 \cos \theta}$

d) $\frac{\cos^3 x}{\cos 2x + \sin^2 x}$

6. Show using a counterexample that the following is not an identity: $\sin (x - y) = \sin x - \sin y$.

7. Simplify $\cos (90° - x)$ using a difference identity.

8. Determine the exact value of each trigonometric expression.

a) $\cos 75°$

b) $\tan 165°$

c) $\sin \frac{7\pi}{12}$

d) $\cos 195°$

e) $\csc \frac{\pi}{12}$

f) $\sin \left(-\frac{\pi}{12}\right)$

Apply

Yukon River at Whitehorse

9. On the winter solstice, December 21 or 22, the power, P, in watts, received from the sun on each square metre of Earth can be determined using the equation $P = 1000 (\sin x \cos 113.5° + \cos x \sin 113.5°)$, where x is the latitude of the location in the northern hemisphere.

a) Use an identity to write the equation in a more useful form.

b) Determine the amount of power received at each location.

i) Whitehorse, Yukon, at 60.7° N

ii) Victoria, British Columbia, at 48.4° N

iii) Igloolik, Nunavut, at 69.4° N

c) Explain the answer for part iii) above. At what latitude is the power received from the sun zero?

10. Simplify $\cos(\pi + x) + \cos(\pi - x)$.

11. Angle θ is in quadrant II and $\sin\theta = \frac{5}{13}$. Determine an exact value for each of the following.

a) $\cos 2\theta$

b) $\sin 2\theta$

c) $\sin\left(\theta + \frac{\pi}{2}\right)$

12. The double-angle identity for tangent in terms of the tangent function is $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$.

a) Verify numerically that this equation is true for $x = \frac{\pi}{6}$.

b) The expression $\tan 2x$ can also be written using the quotient identity for tangent: $\tan 2x = \frac{\sin 2x}{\cos 2x}$. Verify this equation numerically when $x = \frac{\pi}{6}$.

c) The expression $\frac{\sin 2x}{\cos 2x}$ from part b) can be expressed as $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$ using double-angle identities. Show how the expression for $\tan 2x$ used in part a) can also be rewritten in the form $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$.

13. The horizontal distance, d, in metres, travelled by a ball that is kicked at an angle, θ, with the ground is modelled by the formula $d = \frac{2(v_0)^2\sin\theta\cos\theta}{g}$, where v_0 is the initial velocity of the ball, in metres per second, and g is the force of gravity (9.8 m/s^2).

a) Rewrite the formula using a double-angle identity.

b) Determine the angle $\theta \in (0°, 90°)$ that would result in a maximum distance for an initial velocity v_0.

c) Explain why it might be easier to answer part b) with the double-angle version of the formula that you determined in part a).

14. If $(\sin x + \cos x)^2 = k$, then what is the value of $\sin 2x$ in terms of k?

15. Show that each expression can be simplified to $\cos 2x$.

a) $\cos^4 x - \sin^4 x$

b) $\frac{\csc^2 x - 2}{\csc^2 x}$

16. Simplify each expression to the equivalent expression shown.

a) $\frac{1 - \cos 2x}{2}$ $\quad$ $\sin^2 x$

b) $\frac{4 - 8\sin^2 x}{2\sin x\cos x}$ $\quad$ $\frac{4}{\tan 2x}$

17. If the point (2, 5) lies on the terminal arm of angle x in standard position, what is the value of $\cos(\pi + x)$?

18. What value of k makes the equation $\sin 5x\cos x + \cos 5x\sin x = 2\sin kx\cos kx$ true?

19. **a)** If $\cos\theta = \frac{3}{5}$ and $0 < \theta < 2\pi$, determine the value(s) of $\sin\left(\theta + \frac{\pi}{6}\right)$.

b) If $\sin\theta = -\frac{2}{3}$ and $\frac{3\pi}{2} < \theta < 2\pi$, determine the value(s) of $\cos\left(\theta + \frac{\pi}{3}\right)$.

20. If $\angle A$ and $\angle B$ are both in quadrant I, and $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, evaluate each of the following.

a) $\cos(A - B)$

b) $\sin(A + B)$

c) $\cos 2A$

d) $\sin 2A$

Extend

21. Determine the missing primary trigonometric ratio that is required for the expression $\frac{\blacksquare\ \sin 2x}{2 - 2\cos^2 x}$ to simplify to

a) $\cos x$

b) 1

22. Use a double-angle identity for cosine to determine the half-angle formula for cosine, $\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$.

23. a) Graph the curve $y = 4 \sin x - 3 \cos x$. Notice that it resembles a sine function.

b) What are the approximate values of a and c for the curve in the form $y = a \sin (x - c)$, where $0 < c < 90°$?

c) Use the difference identity for sine to rewrite the curve for $y = 4 \sin x - 3 \cos x$ in the form $y = a \sin (x - c)$.

24. Write the following equation in the form $y = A \sin Bx + D$, where A, B, and D are constants:
$y = 6 \sin x \cos^3 x + 6 \sin^3 x \cos x - 3$

Create Connections

C1 a) Determine the value of $\sin 2x$ if $\cos x = -\frac{5}{13}$ and $\pi < x < \frac{3\pi}{2}$ using

i) transformations

ii) a double-angle identity

b) Which method do you prefer? Explain.

C2 a) Graph the function $f(x) = 6 \sin x \cos x$ over the interval $0° \le x \le 360°$.

b) The function can be written as a sine function in the form $f(x) = a \sin bx$. Compare how to determine this sine function from the graph versus using the double-angle identity for sine.

C3 a) Over the domain $0° \le x \le 360°$, sketch the graphs of $y_1 = \sin^2 x$ and $y_2 = \cos^2 x$. How do these graphs compare?

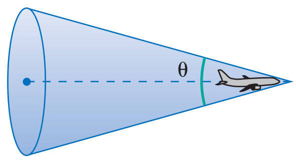
b) Predict what the graph of $y_1 + y_2$ looks like. Explain your prediction. Graph to test your prediction.

c) Graph the difference of the two functions: $y_1 - y_2$. Describe how the two functions interact with each other in the new function.

d) The new function from part c) is sinusoidal. Determine the function in the form $f(x) = a \cos bx$. Explain how you determined the expression.

Project Corner — Mach Numbers

- In aeronautics, the Mach number, M, of an aircraft is the ratio of its speed as it moves through air to the speed of sound in air. An aircraft breaks the sound barrier when its speed is greater than the speed of sound in dry air at 20 °C.
- When an aircraft exceeds Mach 1, $M > 1$, a shock wave forms a cone that spreads backward and outward from the aircraft. The angle at the vertex of a cross-section of the cone is related to the Mach number by $\frac{1}{M} = \sin \frac{\theta}{2}$.



- How could you use the half-angle identity, $\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$, to express the Mach number, M, as a function of θ?
- If plane A is travelling twice as fast as plane B, how are the angles of the cones formed by the planes related?

6.3

Proving Identities

Focus on...

- proving trigonometric identities algebraically
- understanding the difference between verifying and proving an identity
- showing that verifying that the two sides of a potential identity are equal for a given value is insufficient to prove the identity

Many formulas in science contain trigonometric functions. In physics, torque (τ), work (W), and magnetic forces (F_B) can be calculated using the following formulas:

$$\tau = rF \sin \theta \qquad W = F\delta r \cos \theta \qquad F_B = qvB \sin \theta$$

In dynamics, which is the branch of mechanics that deals with motion, trigonometric functions may be required to calculate horizontal and vertical components. Skills with identities reduce the time it takes to work with formulas involving trigonometric functions.

Investigate the Equivalence of Two Trigonometric Expressions

Materials

- graphing calculator

Two physics students are investigating the horizontal distance, d, travelled by a model rocket. The rocket is launched with an angle of elevation θ. Katie has found a formula to model this situation: $d = \dfrac{(v_0)^2 \sin 2\theta}{g}$, where g represents the force of gravity and v_0 represents the initial velocity. Sergey has found a different formula: $d = \dfrac{2(v_0)^2}{g}(\tan \theta - \tan \theta \sin^2 \theta)$.

1. Are the two expressions, $\dfrac{(v_0)^2 \sin 2\theta}{g}$ and $\dfrac{2(v_0)^2}{g}(\tan \theta - \tan \theta \sin^2 \theta)$, equivalent? Use graphical and numerical methods to explain your answer. The initial velocity, v_0, of the rocket is 14 m/s and g is 9.8 m/s^2, so first substitute these values and simplify each expression.
2. Which parts are common to both formulas?
3. Write an identity with the parts of the formulas that are not common. Use your knowledge of identities to rewrite each side and show that they are equivalent.
4. Compare your reasoning with that of a classmate.

Reflect and Respond

5. How does this algebraic method for verifying an identity compare to verifying an identity graphically or numerically? Why do numerical and graphical verification fail to prove that an identity is true?

Link the Ideas

To prove that an identity is true for all permissible values, it is necessary to express both sides of the identity in equivalent forms. One or both sides of the identity must be algebraically manipulated into an equivalent form to match the other side.

You cannot perform operations across the equal sign when proving a potential identity. Simplify the expressions on each side of the identity independently.

Example 1

Verify Versus Prove That an Equation Is an Identity

a) Verify that $1 - \sin^2 x = \sin x \cos x \cot x$ for some values of x. Determine the non-permissible values for x. Work in degrees.

b) Prove that $1 - \sin^2 x = \sin x \cos x \cot x$ for all permissible values of x.

Solution

a) First, determine the non-permissible values.

The only function in the equation that has non-permissible values in its domain is $\cot x$.

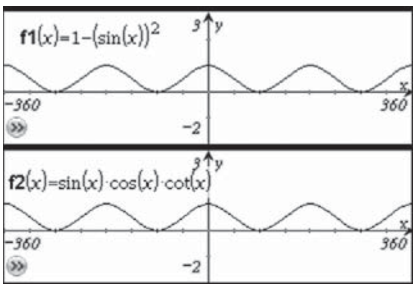
Recall that $\cot x$ is undefined when $\sin x = 0$.

Therefore, $x \neq 180°n$, where $n \in I$.

Verify the identity graphically and numerically.

Method 1: Verify Graphically

Use technology to graph $y = 1 - \sin^2 x$ and $y = \sin x \cos x \cot x$ over the domain $-360° \leq x \leq 360°$. The graphs appear to be the same. So, graphically, it seems that $1 - \sin^2 x = \sin x \cos x \cot x$ is an identity.



Why are the non-permissible values not apparent from these graphs?

Method 2: Verify Numerically

Use $x = 30°$.

Why is 30° a good choice?

$$\begin{aligned}\text{Left Side} &= 1 - \sin^2 x \\ &= 1 - \sin^2 30° \\ &= 1 - \left(\frac{1}{2}\right)^2 \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4}\end{aligned}$$

$$\begin{aligned}\text{Right Side} &= \sin x \cos x \cot x \\ &= \sin 30° \cos 30° \cot 30° \\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{1}\right) \\ &= \frac{3}{4}\end{aligned}$$

Left Side = Right Side

The equation $1 - \sin^2 x = \sin x \cos x \cot x$ is verified for $x = 30°$.

b) To prove the identity algebraically, examine both sides of the equation and simplify each side to a common expression.

$$\begin{aligned}\text{Left Side} &= 1 - \sin^2 x \\ &= \cos^2 x\end{aligned}$$

Why is this true?

$$\begin{aligned}\text{Right Side} &= \sin x \cos x \cot x \\ &= \sin x \cos x \left(\frac{\cos x}{\sin x}\right) \\ &= \cos^2 x\end{aligned}$$

Left Side = Right Side

Therefore, $1 - \sin^2 x = \sin x \cos x \cot x$ is an identity for $x \neq 180°n$, where $n \in \mathrm{I}$.

Your Turn

a) Determine the non-permissible values for the equation $\frac{\tan x \cos x}{\csc x} = 1 - \cos^2 x$.

b) Verify that the equation may be an identity, either graphically using technology or by choosing one value for x.

c) Prove that the identity is true for all permissible values of x.

Example 2

Prove an Identity Using Double-Angle Identities

Prove that $\tan x = \frac{1 - \cos 2x}{\sin 2x}$ is an identity for all permissible values of x.

Solution

$$\text{Left Side} = \tan x$$

$$\begin{aligned}\text{Right Side} &= \frac{1 - \cos 2x}{\sin 2x} \\ &= \frac{1 - (1 - 2\sin^2 x)}{2 \sin x \cos x} \\ &= \frac{2\sin^2 x}{2 \sin x \cos x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x\end{aligned}$$

Recall the double-angle identities.

Remove common factors.

Left Side = Right Side

Therefore, $\tan x = \frac{1 - \cos 2x}{\sin 2x}$ is an identity for all permissible values of x.

Your Turn

Prove that $\frac{\sin 2x}{\cos 2x + 1} = \tan x$ is an identity for all permissible values of x.

In the previous example, you did not need to simplify the left side of the identity. However, tan x could have been expressed as $\frac{\sin x}{\cos x}$ using the quotient identity for tangent. In this case, the right side of the proof would have ended one step earlier, at $\frac{\sin x}{\cos x}$. Sometimes it is advisable to convert all trigonometric functions to expressions of sine or cosine.

Example 3

Prove More Complicated Identities

Prove that $\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$ is an identity for all permissible values of x.

Solution

$$\text{Left Side} = \frac{1 - \cos x}{\sin x}$$

$$\begin{aligned}\text{Right Side} &= \frac{\sin x}{1 + \cos x}\\ &= \frac{\sin x}{1 + \cos x} \times \frac{1 - \cos x}{1 - \cos x}\\ &= \frac{\sin x\,(1 - \cos x)}{1 - \cos^2 x}\\ &= \frac{\sin x\,(1 - \cos x)}{\sin^2 x}\\ &= \frac{1 - \cos x}{\sin x}\end{aligned}$$

How does multiplying by $1 - \cos x$, which is the conjugate of $1 + \cos x$, let you express the denominator in terms of $\sin x$?

Left Side = Right Side

Therefore, $\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$ is an identity for all permissible values of x.

Your Turn

Prove that $\frac{1}{1 + \sin x} = \frac{\sec x - \sin x \sec x}{\cos x}$ is an identity for all permissible values of x.

Example 4

Prove an Identity That Requires Factoring

Prove the identity $\cot x - \csc x = \dfrac{\cos 2x - \cos x}{\sin 2x + \sin x}$ for all permissible values of x.

Solution

$$\begin{aligned}\text{Left Side} &= \cot x - \csc x\\ &= \frac{\cos x}{\sin x} - \frac{1}{\sin x}\\ &= \frac{\cos x - 1}{\sin x}\end{aligned}$$

$$\begin{aligned}\text{Right Side} &= \frac{\cos 2x - \cos x}{\sin 2x + \sin x}\\ &= \frac{(2\cos^2 x - 1) - \cos x}{2\sin x\cos x + \sin x}\\ &= \frac{2\cos^2 x - \cos x - 1}{\sin x\,(2\cos x + 1)}\\ &= \frac{(2\cos x + 1)(\cos x - 1)}{\sin x\,(2\cos x + 1)}\\ &= \frac{\cos x - 1}{\sin x}\end{aligned}$$

Why is $2\cos^2 x - 1$ substituted for $\cos 2x$?

Left Side = Right Side

Therefore, $\cot x - \csc x = \dfrac{\cos 2x - \cos x}{\sin 2x + \sin x}$ is an identity for all permissible values of x.

Your Turn

Prove the identity $\dfrac{\sin 2x - \cos x}{4\sin^2 x - 1} = \dfrac{\sin^2 x\cos x + \cos^3 x}{2\sin x + 1}$ for all permissible values of x.

Key Ideas

- Verifying an identity using a specific value validates that it is true for that value only. Proving an identity is done algebraically and validates the identity for all permissible values of the variable.
- To prove a trigonometric identity algebraically, separately simplify both sides of the identity into identical expressions.
- It is usually easier to make a complicated expression simpler than it is to make a simple expression more complicated.
- Some strategies that may help you prove identities include:
  - Use known identities to make substitutions.
  - If quadratics are present, the Pythagorean identity or one of its alternate forms can often be used.
  - Rewrite the expression using sine and cosine only.
  - Multiply the numerator and the denominator by the conjugate of an expression.
  - Factor to simplify expressions.

Check Your Understanding

Practise

1. Factor and simplify each rational trigonometric expression.
   a) $\dfrac{\sin x - \sin x \cos^2 x}{\sin^2 x}$
   b) $\dfrac{\cos^2 x - \cos x - 2}{6 \cos x - 12}$
   c) $\dfrac{\sin x \cos x - \sin x}{\cos^2 x - 1}$
   d) $\dfrac{\tan^2 x - 3 \tan x - 4}{\sin x \tan x + \sin x}$

2. Use factoring to help to prove each identity for all permissible values of x.
   a) $\cos x + \cos x \tan^2 x = \sec x$
   b) $\dfrac{\sin^2 x - \cos^2 x}{\sin x + \cos x} = \sin x - \cos x$
   c) $\dfrac{\sin x \cos x - \sin x}{\cos^2 x - 1} = \dfrac{1 - \cos x}{\sin x}$
   d) $\dfrac{1 - \sin^2 x}{1 + 2 \sin x - 3 \sin^2 x} = \dfrac{1 + \sin x}{1 + 3 \sin x}$

3. Use a common denominator to express the rational expressions as a single term.
   a) $\dfrac{\sin x}{\cos x} + \sec x$
   b) $\dfrac{1}{\sin x - 1} + \dfrac{1}{\sin x + 1}$
   c) $\dfrac{\sin x}{1 + \cos x} + \dfrac{\cos x}{\sin x}$
   d) $\dfrac{\cos x}{\sec x - 1} + \dfrac{\cos x}{\sec x + 1}$

4. a) Rewrite the expression $\dfrac{\sec x - \cos x}{\tan x}$ in terms of sine and cosine functions only.
   b) Simplify the expression to one of the primary trigonometric functions.

5. Verify graphically that $\cos x = \dfrac{\sin 2x}{2 \sin x}$ could be an identity. Then, prove the identity. Determine any non-permissible values.

6. Expand and simplify the expression $(\sec x - \tan x)(\sin x + 1)$ to a primary trigonometric function.

7. Prove each identity.
   a) $\dfrac{\csc x}{2 \cos x} = \csc 2x$
   b) $\sin x + \cos x \cot x = \csc x$

Apply

8. As the first step of proving the identity $\dfrac{\cos 2x - 1}{\sin 2x} = -\tan x$, Hanna chose to substitute $\cos 2x = 1 - 2 \sin^2 x$, while Chloe chose $\cos 2x = 2 \cos^2 x - 1$. Which choice leads to a shorter proof? Explain. Prove the identity.

9. The distance, d, in metres, that a golf ball travels when struck by a golf club is given by the formula $d = \dfrac{(v_0)^2 \sin 2\theta}{g}$, where v_0 is the initial velocity of the ball, θ is the angle between the ground and the initial path of the ball, and g is the acceleration due to gravity (9.8 m/s²).

   a) What distance, in metres, does the ball travel if its initial velocity is 21 m/s and the angle θ is 55°?
   b) Prove the identity $\dfrac{(v_0)^2 \sin 2\theta}{g} = \dfrac{2(v_0)^2(1 - \cos^2 \theta)}{g \tan \theta}$.

10. Verify each potential identity by graphing, and then prove the identity.
   a) $\dfrac{\csc x}{2 \cos x} = \csc 2x$
   b) $\dfrac{\sin x \cos x}{1 + \cos x} = \dfrac{1 - \cos x}{\tan x}$
   c) $\dfrac{\sin x + \tan x}{1 + \cos x} = \dfrac{\sin 2x}{2 \cos^2 x}$

11. Prove each identity.
   a) $\dfrac{\sin 2x}{\cos x} + \dfrac{\cos 2x}{\sin x} = \csc x$
   b) $\csc^2 x + \sec^2 x = \csc^2 x \sec^2 x$
   c) $\dfrac{\cot x - 1}{1 - \tan x} = \dfrac{\csc x}{\sec x}$

12. Prove each identity.

a) $\sin(90° + \theta) = \sin(90° - \theta)$

b) $\sin(2\pi - \theta) = -\sin\theta$

13. Prove that
$2\cos x\cos y = \cos(x + y) + \cos(x - y)$.

14. Consider the equation
$\cos 2x = 2\sin x\cos x$.

a) Graph each side of the equation. Could the equation be an identity?

b) Either prove that the equation is an identity or find a counterexample to show that it is not an identity.

15. Consider the equation $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$.

a) Determine the non-permissible values for x.

b) Prove that the equation is an identity for all permissible values of x.

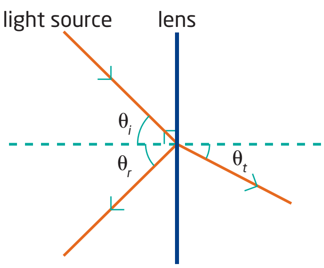
Extend

16. Use double-angle identities to prove the identity $\tan x = \dfrac{\sin 4x - \sin 2x}{\cos 4x + \cos 2x}$.

17. Verify graphically and then prove the identity $\dfrac{\sin 2x}{1 - \cos 2x} = 2\csc 2x - \tan x$.

18. Prove the identity
$\dfrac{1 - \sin^2 x - 2\cos x}{\cos^2 x - \cos x - 2} = \dfrac{1}{1 + \sec x}$.

19. When a ray of light hits a lens at angle of incidence θ_i, some of the light is refracted (bent) as it passes through the lens, and some is reflected by the lens. In the diagram, θ_r is the angle of reflection and θ_t is the angle of refraction. Fresnel equations describe the behaviour of light in this situation.



a) Snells's law states that $n_1 \sin\theta_i = n_2 \sin\theta_t$, where n_1 and n_2 are the refractive indices of the mediums. Isolate $\sin\theta_t$ in this equation.

b) Under certain conditions, a Fresnel equation to find the fraction, R, of light reflected is
$$R = \left(\frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}\right)^2.$$
Use identities to prove that this can be written as
$$R = \left(\frac{n_1\cos\theta_i - n_2\sqrt{1 - \sin^2\theta_t}}{n_1\cos\theta_i + n_2\sqrt{1 - \sin^2\theta_t}}\right)^2.$$

c) Use your work from part a) to prove that
$$\left(\frac{n_1\cos\theta_i - n_2\sqrt{1 - \sin^2\theta_t}}{n_1\cos\theta_i + n_2\sqrt{1 - \sin^2\theta_t}}\right)^2 = \left(\frac{n_1\cos\theta_i - n_2\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta_i}}{n_1\cos\theta_i + n_2\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta_i}}\right)^2$$

Did You Know?

Fresnel equations were developed by French physicist Augustin-Jean Fresnel (1788–1827). A Fresnel lens is also named for him, and is a common lens in lights used for movies, TV, and live theatre. A new use for Fresnel lenses is to focus light in a solar array to allow for much more efficient collection of solar energy.

Create Connections

C1 Why is verifying, either numerically or graphically, that both sides of an equation seem to be equal not sufficient to prove that the equation is an identity?

C2 Use the difference identity for cosine to prove the identity $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

C3 Consider the equation
$\cos x = \sqrt{1 - \sin^2 x}$.

a) What are the non-permissible values for x in this equation?

b) What is a value for x that makes this equation true?

c) What is a value for x that does not work in this equation and provides evidence that this equation is not an identity?

d) Explain the difference between an identity and an equation.

6.4

Solving Trigonometric Equations Using Identities

Focus on...

- solving trigonometric equations algebraically using known identities
- determining exact solutions for trigonometric equations where possible
- determining the general solution for trigonometric equations
- identifying and correcting errors in a solution for a trigonometric equation

Sound from a musical instrument is composed of sine waves. Technicians often fade the sound near the end of a song. To create this effect, the sound equipment is programmed to use mathematical damping techniques. The technicians have three choices: a linear fade, a logarithmic fade, or an inverse logarithmic fade. You will explore logarithmic functions in Chapter 8.

Knowledge of trigonometric identities can help to simplify the expressions involved in the trigonometric equations of sound waves in music.

Did You Know?

The musical instrument with the purest sound wave is the flute. The most complex musical sound wave can be created with a cymbal.

Investigate Solving Trigonometric Equations

Materials

- graphing technology

1. Graph the function $y = \sin 2x - \sin x$ over the domain $-720° < x \leq 720°$. Make a sketch of the graph and describe it in words.
2. From the graph, determine an expression for the zeros of the function $y = \sin 2x - \sin x$ over the domain of all real numbers.
3. Algebraically solve the equation $\sin 2x - \sin x = 0$ over the domain of all real numbers. Compare your answer and method with those of a classmate.

Reflect and Respond

4. Which method, graphic or algebraic, do you prefer to solve the equation $\sin 2x - \sin x = 0$? Explain.

Link the Ideas

To solve some trigonometric equations, you need to make substitutions using the trigonometric identities that you have studied in this chapter. This often involves ensuring that the equation is expressed in terms of one trigonometric function.

Example 1

Solve by Substituting Trigonometric Identities and Factoring

Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $\cos 2x + 1 - \cos x = 0$

b) $1 - \cos^2 x = 3 \sin x - 2$

Solution

a)

$$\cos 2x + 1 - \cos x = 0$$

$$(2\cos^2 x - 1) + 1 - \cos x = 0$$

Why is this version of the identity for $\cos 2x$ chosen?

$$2\cos^2 x - \cos x = 0$$

Simplify.

$$\cos x\,(2\cos x - 1) = 0$$

Factor.

$$\cos x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$

Use the zero product property.

$$x = \frac{\pi}{2} \text{ or } x = \frac{3\pi}{2} \qquad \text{or } \cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3} \text{ or } x = \frac{5\pi}{3}$$

There are no non-permissible values for the original equation, so the solutions over the domain $0 \leq x < 2\pi$ are $x = \frac{\pi}{3}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and $x = \frac{5\pi}{3}$.

b)

$$1 - \cos^2 x = 3\sin x - 2$$

$$\sin^2 x = 3\sin x - 2$$

Use the Pythagorean identity.

$$\sin^2 x - 3\sin x + 2 = 0$$

$$(\sin x - 1)(\sin x - 2) = 0$$

Use the zero product property.

$$\sin x - 1 = 0 \text{ or } \qquad \sin x - 2 = 0$$

$$\sin x = 1 \qquad \sin x = 2$$

$$x = \frac{\pi}{2} \qquad \sin x = 2 \text{ has no solution.}$$

Why is there no solution for $\sin x = 2$?

There are no non-permissible values for the original equation, so the solution over the domain $0 \leq x < 2\pi$ is $x = \frac{\pi}{2}$.

Your Turn

Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $\sin 2x - \cos x = 0$

b) $2\cos x + 1 - \sin^2 x = 3$

Example 2

Solve an Equation With a Quotient Identity Substitution

a) Solve the equation $\cos^2 x = \cot x \sin x$ algebraically in the domain $0° \leq x < 360°$.

b) Verify your answer graphically.

Solution

a)

$$\cos^2 x = \cot x \sin x$$

What is the quotient identity for cot *x*?

$$\cos^2 x = \left(\frac{\cos x}{\sin x}\right) \sin x$$

$$\cos^2 x = \cos x$$

Why is it incorrect to divide by cos *x* here?

$$\cos^2 x - \cos x = 0$$

$$\cos x (\cos x - 1) = 0$$

Factor.

$$\cos x = 0 \text{ or } \cos x = 1$$

Apply the zero product property.

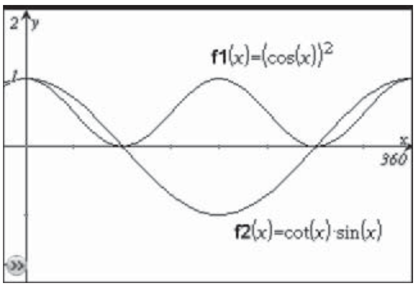
For $\cos x = 0$, $x = 90°$ and $x = 270°$.
For $\cos x = 1$, $x = 0°$.

Check whether there are any non-permissible values for the initial equation.

For $\cot x$, the domain has the restriction $\sin x \neq 0$, which gives the non-permissible values $x \neq 0°$ and $x \neq 180°$.

Therefore, the solution for $\cos^2 x = \cot x \sin x$ is limited to $x = 90°$ and $x = 270°$.

b) Graph $y = \cos^2 x$ and $y = \cot x \sin x$ over the domain $0° \leq x < 360°$. Determine the points of intersection of the two functions.



It appears from the graph that a solution is $x = 0$. Note that $y = \cot x \sin x$ is not defined at $x = 0$ because it is a non-permissible value for $\cot x$.

What solutions are confirmed by the graph?

Your Turn

a) Solve the equation $\sin^2 x = \frac{1}{2} \tan x \cos x$ algebraically over the domain $0° \leq x < 360°$.

b) Verify your answer graphically.

Example 3

Determine the General Solution for a Trigonometric Equation

Solve the equation $\sin 2x = \sqrt{2} \cos x$ algebraically. Give the general solution expressed in radians.

Solution

$$\sin 2x = \sqrt{2} \cos x$$

$$2 \sin x \cos x = \sqrt{2} \cos x$$ Use the double-angle identity for $\sin 2x$.

$$2 \sin x \cos x - \sqrt{2} \cos x = 0$$

$$\cos x\,(2 \sin x - \sqrt{2}) = 0$$ Why is it incorrect to divide by $\cos x$ here?

Then, $\cos x = 0$ or $2 \sin x - \sqrt{2} = 0$

$$\sin x = \frac{\sqrt{2}}{2}$$

For $\cos x = 0$, $x = \frac{\pi}{2} + \pi n$, where $n \in I$.

For $\sin x = \frac{\sqrt{2}}{2}$, $x = \frac{\pi}{4} + 2\pi n$ and $x = \frac{3\pi}{4} + 2\pi n$, where $n \in I$.

Since there are no non-permissible values for the original equation, the solution is $x = \frac{\pi}{2} + \pi n$, $x = \frac{\pi}{4} + 2\pi n$, and $x = \frac{3\pi}{4} + 2\pi n$, where $n \in I$.

Your Turn

Algebraically solve $\cos 2x = \cos x$. Give general solutions expressed in radians.

Example 4

Determine the General Solution Using Reciprocal Identities

Algebraically solve $2 \sin x = 7 - 3 \csc x$. Give general solutions expressed in radians.

Solution

$$2 \sin x = 7 - 3 \csc x$$

$$2 \sin x = 7 - \frac{3}{\sin x}$$ Use the reciprocal identity for cosecant.

$$\sin x\,(2 \sin x) = \sin x\left(7 - \frac{3}{\sin x}\right)$$ Why multiply both sides by $\sin x$?

$$2 \sin^2 x = 7 \sin x - 3$$

$$2 \sin^2 x - 7 \sin x + 3 = 0$$

$$(2 \sin x - 1)(\sin x - 3) = 0$$ Factor.

For $2 \sin x - 1 = 0$, Use the zero product property.

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} + 2\pi n \text{ and } x = \frac{5\pi}{6} + 2\pi n$$

For $\sin x - 3 = 0$,

$$\sin x = 3$$ Why is there no solution for $\sin x = 3$?

There is no solution for $\sin x = 3$.

The restriction on the original equation is $\sin x \neq 0$ because of the presence of $\csc x$.

Since $\sin x = 0$ does not occur in the solution, all determined solutions are permissible.

The solution is $x = \frac{\pi}{6} + 2\pi n$ and $x = \frac{5\pi}{6} + 2\pi n$, where $n \in \mathrm{I}$.

Your Turn

Algebraically solve $3 \cos x + 2 = 5 \sec x$. Give general solutions expressed in radians.

Key Ideas

- Reciprocal, quotient, Pythagorean, and double-angle identities can be used to help solve a trigonometric equation algebraically.
- The algebraic solution for a trigonometric equation can be verified graphically.
- Check that solutions for an equation do not include non-permissible values from the original equation.
- Unless the domain is restricted, give general solutions. For example, for $2 \cos x = 1$, the general solution is $x = \frac{\pi}{3} + 2\pi n$ and $x = \frac{5\pi}{3} + 2\pi n$, where $n \in \mathrm{I}$. If the domain is specified as $0° \leq x < 360°$, then the solutions are $60°$ and $300°$.

Check Your Understanding

Practise

1. Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $\tan^2 x - \tan x = 0$

b) $\sin 2x - \sin x = 0$

c) $\sin^2 x - 4 \sin x = 5$

d) $\cos 2x = \sin x$

2. Solve each equation algebraically over the domain $0° \leq x < 360°$. Verify your solution graphically.

a) $\cos x - \cos 2x = 0$

b) $\sin^2 x - 3 \sin x = 4$

c) $\tan x \cos x \sin x - 1 = 0$

d) $\tan^2 x + \sqrt{3} \tan x = 0$

3. Rewrite each equation in terms of sine only. Then, solve algebraically for $0 \leq x < 2\pi$.

a) $\cos 2x - 3 \sin x = 2$

b) $2 \cos^2 x - 3 \sin x - 3 = 0$

c) $3 \csc x - \sin x = 2$

d) $\tan^2 x + 2 = 0$

4. Solve $4 \sin^2 x = 1$ algebraically over the domain $-180° \leq x < 180°$.

5. Solve $2 \tan^2 x + 3 \tan x - 2 = 0$ algebraically over the domain $0 \leq x < 2\pi$.

Apply

6. Determine the mistake that Sanesh made in the following work. Then, complete a correct solution.

Solve $2\cos^2 x = \sqrt{3}\cos x$. Express your answer(s) in degrees.

Solution:

$$\frac{1}{\cos x}(2\cos^2 x) = (\sqrt{3}\cos x)\frac{1}{\cos x}$$

$$2\cos x = \sqrt{3}$$

$$\cos x = \frac{\sqrt{3}}{2}$$

$$x = 30^\circ + 360^\circ n \text{ and } x = 330^\circ + 360^\circ n$$

7. a) Solve algebraically $\sin 2x = 0.5$, $0 \le x < 2\pi$.

b) Solve the equation from part a) using a different method.

8. Solve $\sin^2 x = \cos^2 x + 1$ algebraically for all values of x. Give your answer(s) in radians.

9. Solve $\cos x \sin 2x - 2\sin x = -2$ algebraically over the domain of real numbers. Give your answer(s) in radians.

10. How many solutions does the equation $(7\sin x + 2)(3\cos x + 3)(\tan^2 x - 2) = 0$ have over the interval $0° < x \le 360°$? Explain your reasoning.

11. Solve $\sqrt{3}\cos x \csc x = -2\cos x$ for x over the domain $0 \le x < 2\pi$.

12. If $\cos x = \frac{2}{3}$ and $\cos x = -\frac{1}{3}$ are the solutions for a trigonometric equation, what are the values of B and C if the equation is of the form $9\cos^2 x + B\cos x + C = 0$?

13. Create a trigonometric equation that includes $\sin 2x$ and that can be solved by factoring. Then, solve it.

14. Solve $\sin 2x = 2\cos x \cos 2x$ algebraically. Give the general solution expressed in radians.

15. Algebraically determine the number of solutions for the equation $\cos 2x \cos x - \sin 2x \sin x = 0$ over the domain $-360° < x \le 360°$.

16. Solve $\sec x + \tan^2 x - 3\cos x = 2$ algebraically. Give the general solution expressed in radians.

Extend

17. Solve $4\sin^2 x = 3\tan^2 x - 1$ algebraically. Give the general solution expressed in radians.

18. Solve $\dfrac{1 - \sin^2 x - 2\cos x}{\cos^2 x - \cos x - 2} = -\dfrac{1}{3}$ algebraically over the domain $-\pi \le x \le \pi$.

19. Find the general solution for the equation $4(16^{\cos^2 x}) = 2^{6\cos x}$. Give your answer in radians.

20. For some angles α and β, $\sin^2 \alpha + \cos^2 \beta = m^2$ and $\cos^2 \alpha + \sin^2 \beta = m$. Find the possible value(s) for m.

Create Connections

C1 Refer to the equation $\sin x - \cos 2x = 0$ to answer the following.

a) Which identity would you use to express the equation in terms of one trigonometric function?

b) How can you solve the resulting equation by factoring?

c) What is the solution for the domain $0° \le x < 360°$?

d) Verify your solution by graphing.

C2 Refer to the equation $3\cos^2 x + \cos x - 1 = 0$ to answer the following.

a) Why is not possible to factor the left side of the equation?

b) Solve the equation using the quadratic formula.

c) What is the solution over the domain $0° \le x < 720°$?

C3 Use the double-angle identity for sine to create an equation that is not an identity. Solve the equation and explain why it is not an identity.

Chapter 6 Review

6.1 Reciprocal, Quotient, and Pythagorean Identities, pages 290–298

1. Determine the non-permissible values, in radians, for each expression.

a) $\dfrac{3 \sin x}{\cos x}$

b) $\dfrac{\cos x}{\tan x}$

c) $\dfrac{\sin x}{1 - 2 \cos x}$

d) $\dfrac{\cos x}{\sin^2 x - 1}$

2. Simplify each expression to one of the three primary trigonometric functions.

a) $\dfrac{\sin x}{\tan x}$

b) $\dfrac{\sec x}{\csc x}$

c) $\dfrac{\sin x + \tan x}{1 + \cos x}$

d) $\dfrac{\csc x - \sin x}{\cot x}$

3. Rewrite each trigonometric expression in terms of sine or cosine or both. Then, simplify.

a) $\tan x \cot x$

b) $\dfrac{1}{\csc^2 x} + \dfrac{1}{\sec^2 x}$

c) $\sec^2 x - \tan^2 x$

4. a) Verify that the potential identity $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$ is true for $x = 30°$ and for $x = \dfrac{\pi}{4}$.

b) What are the non-permissible values for the equation over the domain $0° \leq x < 360°$?

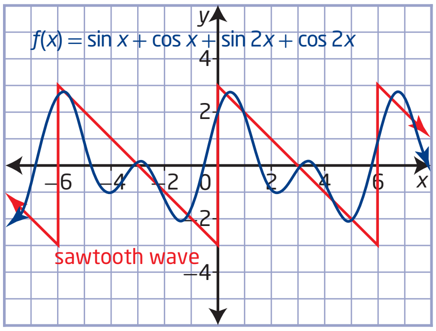
5. a) Determine two values of x that satisfy the equation $\sqrt{\tan^2 x + 1} = \sec x$.

b) Use technology to graph $y = \sqrt{\tan^2 x + 1}$ and $y = \sec x$ over the domain $-\dfrac{\pi}{2} \leq x < \dfrac{3\pi}{2}$. Compare the two graphs.

c) Explain, using your graph in part b), how you know that $\sqrt{\tan^2 x + 1} = \sec x$ is not an identity.

6.2 Sum, Difference, and Double-Angle Identities, pages 299–308

6. A Fourier series is an infinite series in which the terms are made up of sine and cosine ratios. A finite number of terms from a Fourier series is often used to approximate the behaviour of waves.



The first four terms of the Fourier series approximation for a sawtooth wave are $f(x) = \sin x + \cos x + \sin 2x + \cos 2x$.

a) Determine the value of $f(0)$ and of $f\left(\dfrac{\pi}{6}\right)$.

b) Prove that $f(x)$ can be written as $f(x) = \sin x + \cos x + 2 \sin x \cos x - 2 \sin^2 x + 1$.

c) Is it possible to rewrite this Fourier series using only sine or only cosine? Justify your answer.

d) Use the pattern in the first four terms to write $f(x)$ with more terms. Graph $y = f(x)$ using technology, for $x \in [-4\pi, 4\pi]$. How many terms are needed to arrive at a good approximation of a sawtooth wave?

7. Write each expression as a single trigonometric function, and then evaluate.

a) $\sin 25° \cos 65° + \cos 25° \sin 65°$

b) $\sin 54° \cos 24° - \cos 54° \sin 24°$

c) $\cos \dfrac{\pi}{4} \cos \dfrac{\pi}{12} + \sin \dfrac{\pi}{4} \sin \dfrac{\pi}{12}$

d) $\cos \dfrac{\pi}{6} \cos \dfrac{\pi}{12} - \sin \dfrac{\pi}{6} \sin \dfrac{\pi}{12}$

8. Use sum or difference identities to find the exact value of each trigonometric expression.

a) $\sin 15°$

b) $\cos\left(-\frac{\pi}{12}\right)$

c) $\tan 165°$

d) $\sin \frac{5\pi}{12}$

9. If $\cos A = -\frac{5}{13}$, where $\frac{\pi}{2} \le A < \pi$, evaluate each of the following.

a) $\cos\left(A - \frac{\pi}{4}\right)$

b) $\sin\left(A + \frac{\pi}{3}\right)$

c) $\sin 2A$

10. What is the exact value of $\left(\sin \frac{\pi}{8} + \cos \frac{\pi}{8}\right)^2$?

11. Simplify the expression $\frac{\cos^2 x - \cos 2x}{0.5 \sin 2x}$ to one of the primary trigonometric ratios.

6.3 Proving Identities, pages 309–315

12. Factor and simplify each expression.

a) $\frac{1 - \sin^2 x}{\cos x \sin x - \cos x}$

b) $\tan^2 x - \cos^2 x \tan^2 x$

13. Prove that each identity holds for all permissible values of x.

a) $1 + \cot^2 x = \csc^2 x$

b) $\tan x = \csc 2x - \cot 2x$

c) $\sec x + \tan x = \frac{\cos x}{1 - \sin x}$

d) $\frac{1}{1 + \cos x} + \frac{1}{1 - \cos x} = 2 \csc^2 x$

14. Consider the equation $\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}$.

a) Verify that the equation is true when $x = \frac{\pi}{4}$. Does this mean that the equation is an identity? Why or why not?

b) What are the non-permissible values for the equation?

c) Prove that the equation is an identity for all permissible values of x.

15. Prove each identity.

a) $\frac{\cos x + \cot x}{\sec x + \tan x} = \cos x \cot x$

b) $\sec x + \tan x = \frac{\cos x}{1 - \sin x}$

16. Consider the equation $\cos 2x = 2 \sin x \sec x$.

a) Describe two methods that can be used to determine whether this equation could be an identity.

b) Use one of the methods to show that the equation is not an identity.

6.4 Solving Trigonometric Equations Using Identities, pages 316–321

17. Solve each equation algebraically over the domain $0 \le x < 2\pi$.

a) $\sin 2x + \sin x = 0$

b) $\cot x + \sqrt{3} = 0$

c) $2 \sin^2 x - 3 \sin x - 2 = 0$

d) $\sin^2 x = \cos x - \cos 2x$

18. Solve each equation algebraically over the domain $0° \le x < 360°$. Verify your solution graphically.

a) $2 \sin 2x = 1$

b) $\sin^2 x = 1 + \cos^2 x$

c) $2 \cos^2 x = \sin x + 1$

d) $\cos x \tan x - \sin^2 x = 0$

19. Algebraically determine the general solution to the equation $4 \cos^2 x - 1 = 0$. Give your answer in radians.

20. If $0° \le x < 360°$, what is the value of $\cos x$ in the equation $2 \cos^2 x + \sin^2 x = \frac{41}{25}$?

21. Use an algebraic approach to find the solution of $2 \sin x \cos x = 3 \sin x$ over the domain $-2\pi \le x \le 2\pi$.

Chapter 6 Practice Test

Multiple Choice

For #1 to #6, choose the best answer.

1. Which expression is equivalent to $\frac{\cos 2x - 1}{\sin 2x}$?

A $-\tan x$ **B** $-\cot x$

C $\tan x$ **D** $\cot x$

2. Which expression is equivalent to $\cot \theta + \tan \theta$?

A $\frac{1}{\sin \theta \cos \theta}$ **B** $\frac{\cos \theta + \sin \theta}{\sin \theta \cos \theta}$

C 1 **D** 2

3. Which expression is equivalent to $\tan^2 \theta \csc \theta + \frac{1}{\sin \theta}$?

A $\sec^3 \theta$ **B** $\csc^3 \theta$

C $\csc^2 \theta \sec \theta$ **D** $\sec^2 \theta \csc \theta$

4. Which single trigonometric function is equivalent to $\cos \frac{\pi}{5} \cos \frac{\pi}{6} - \sin \frac{\pi}{5} \sin \frac{\pi}{6}$?

A $\cos \frac{\pi}{30}$ **B** $\sin \frac{\pi}{30}$

C $\sin \frac{11\pi}{30}$ **D** $\cos \frac{11\pi}{30}$

5. Simplified, $4 \cos^2 x - 2$ is equivalent to

A $2 \cos 2x$ **B** $4 \cos 2x$

C $2 \cos 4x$ **D** $4 \cos x$

6. If $\sin \theta = c$ and $0 \le \theta < \frac{\pi}{2}$, which expression is equivalent to $\cos (\pi + \theta)$?

A $1 - c$ **B** $c - 1$

C $\sqrt{1 - c^2}$ **D** $-\sqrt{1 - c^2}$

Short Answer

7. Determine the exact value of each trigonometric ratio.

a) $\cos 105°$ **b)** $\sin \frac{5\pi}{12}$

8. Prove the identity $\cot \theta - \tan \theta = 2 \cot 2\theta$. Determine the non-permissible value(s), if any.

9. In physics, two students are doing a report on the intensity of light passing through a filter. After some research, they each find a different formula for the situation. In each formula, I is the intensity of light passing through the filter, I_0 is the initial light shining on the filter, θ is the angle between the axes of polarization.

Theo's formula: $I = I_0 \cos^2 \theta$

Sany's formula: $I = I_0 - \frac{I_0}{\csc^2 \theta}$

Prove that the two formulas are equivalent.

10. Determine the general solution, in radians, for each equation.

a) $\sec A + 2 = 0$

b) $2 \sin B = 3 \tan^2 B$

c) $\sin 2\theta \sin \theta + \cos^2 \theta = 1$

11. Solve the equation $\sin 2x + 2 \cos x = 0$ algebraically. Give the general solution in radians.

12. If $\sin \theta = -\frac{4}{5}$ and θ is in quadrant III, determine the exact value(s) of $\cos \left(\theta - \frac{\pi}{6}\right)$.

13. Solve $2 \tan x \cos^2 x = 1$ algebraically over the domain $0 \le x < 2\pi$.

Extended Response

14. Solve $\sin^2 x + \cos 2x - \cos x = 0$ over the domain $0° \le x < 360°$. Verify your solution graphically.

15. Prove each identity.

a) $\frac{\cot x}{\csc x - 1} = \frac{\csc x + 1}{\cot x}$

b) $\sin (x + y) \sin (x - y) = \sin^2 x - \sin^2 y$

16. Algebraically find the general solution for $2 \cos^2 x + 3 \sin x - 3 = 0$. Give your answer in radians.

Unit 2 Project Wrap-Up

Applications of Trigonometry

You can describe the relationships between angles, trigonometric ratios, and the unit circle. You can graph and analyse trigonometric functions and transformations of sinusoidal functions. You can identify and prove trigonometric identities. This knowledge will help you solve problems using trigonometric ratios and equations.

Complete at least one of the following options:

Option 1 Angle Measure

Research the types of units for angle measure and their history.

- Search for when, why, who, where, and what, relating to degrees, radians, and other types of units for angular measure.
- Prepare a presentation or report discussing the following:
  - Why was radian measure invented?
  - Why is π used for radian measure?
  - Which unit of measure do you prefer? Explain.
  - Why do other types of units for angle measure exist? In which situations are they used?

Option 2 Broadcasting

Research periodic functions as they relate to broadcasting.

- Search the Internet for carrier waveforms—what they are, how they work, and their connection to periodic functions.
- Prepare a presentation or report including the following:
  - a brief description of carrier waveforms and their significance
  - an example of carrier waveforms in use, including a diagram
  - an explanation of the mathematics involved and how it helps to model a broadcast

Option 3 Mach Numbers

Search for information relating supersonic travel and trigonometry.

- Prepare a presentation or report including the following:
  - a brief description of Mach numbers and an explanation of the mathematics involved in expressing a Mach number as a function of θ
  - examples of Mach numbers and resulting shock wave cones
  - an explanation of the effects of increasing Mach numbers on the cone angle, θ

Option 4 Crime Scene Investigation

Research how trigonometry and trigonometric functions are used to analyse crime scenes.

- Prepare a report that addresses at least two areas that are used by forensic scientists to solve and piece together the events of a crime scene.
  - Some choices are trajectory determination, blood pattern identification, and background sound analysis from videotapes or cell phones.
- Identify and explain the trigonometric functions used, one of which must be a sine function, and what the variables represent.
- Show the application to a problem by providing the calculations and interpreting the results.

Cumulative Review, Chapters 4–6

Chapter 4 Trigonometry and the Unit Circle

1. Draw each angle in standard position. Write an expression for all angles that are coterminal with each given angle.

a) $\frac{7\pi}{3}$ **b)** $-100°$

2. Convert each radian measure to degrees. Express your answers to the nearest degree.

a) 4 **b)** $\frac{-5\pi}{3}$

3. Convert each degree measure to radians. Express your answers as exact values.

a) 210° **b)** $-500°$

4. A Ferris wheel that is 175 ft in diameter has 42 gondolas.

a) Determine the arc length, to the nearest tenth of a foot, between each gondola.

b) When the first gondola rotates through 70°, determine the distance it travels, to the nearest tenth of a foot.

5. Determine the equation of a circle centred at the origin

a) with a radius of 5

b) if the point $P(3, \sqrt{7})$ is on the circle

6. $P(\theta)$ is the point where the terminal arm of an angle θ intersects the unit circle. If $P(\theta) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, complete the following.

a) In which quadrant does θ terminate?

b) Determine all measures for θ in the interval $-2\pi \leq \theta \leq 2\pi$.

c) What are the coordinates of $P\left(\theta + \frac{\pi}{2}\right)$? Explain how you know.

d) What are the coordinates of $P(\theta - \pi)$? Explain how you know.

7. $P(\theta)$ is the point where the terminal arm of an angle θ intersects the unit circle.

a) Determine the coordinates of $P(-45°)$ and $P(45°)$. How are the answers related?

b) Determine the coordinates of $P(675°)$ and $P(765°)$. How are the answers related?

8. Determine the exact value of each trigonometric ratio.

a) $\sin \frac{4\pi}{3}$ **b)** $\cos 300°$

c) $\tan (-570°)$ **d)** $\csc 135°$

e) $\sec \left(-\frac{3\pi}{2}\right)$ **f)** $\cot \frac{23\pi}{6}$

9. The terminal arm of an angle θ in standard position passes through the point $P(-9, 12)$.

a) Draw the angle in standard position.

b) Determine the exact values of the six trigonometric ratios.

c) Determine the approximate measure of all possible values of θ, to the nearest hundredth of a degree.

10. Determine the exact roots for each equation.

a) $\sin \theta = -\frac{1}{2}$, $-2\pi \leq \theta \leq 2\pi$

b) $\sec \theta = \frac{2\sqrt{3}}{3}$, $-180° \leq \theta \leq 180°$

c) $\tan \theta = -1$, $0 \leq \theta \leq 2\pi$

11. Determine the general solution, in radians, for each equation.

a) $\cos \theta = -\frac{\sqrt{2}}{2}$ **b)** $\csc \theta = 1$

c) $\cot \theta = 0$

12. Solve each equation over the domain $0 \leq \theta \leq 2\pi$. Express your answers as exact values.

a) $\sin \theta = \sin \theta \tan \theta$

b) $2 \cos^2 \theta + 5 \cos \theta + 2 = 0$

13. Solve for θ, where $0° \leq \theta \leq 360°$. Give your answers as approximate measures, to the nearest degree.

a) $4 \tan^2 \theta - 1 = 0$

b) $3 \sin^2 \theta - 2 \sin \theta = 1$

Chapter 5 Trigonometric Functions and Graphs

14. Write the equation of the sine function with amplitude 3, period 4π, and phase shift $\frac{\pi}{4}$ units to the left.

15. Determine the amplitude, period, phase shift, and vertical displacement with respect to $y = \sin \theta$ or $y = \cos \theta$ for each function. Sketch the graph of each function for two cycles.

a) $y = 3 \cos 2\theta$

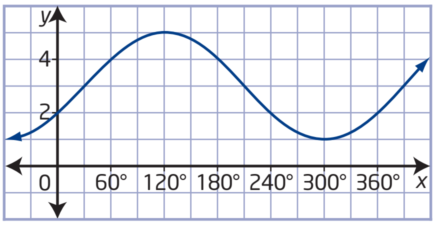
b) $y = -2 \sin (3\theta + 60°)$

c) $y = \frac{1}{2} \cos (\theta + \pi) - 4$

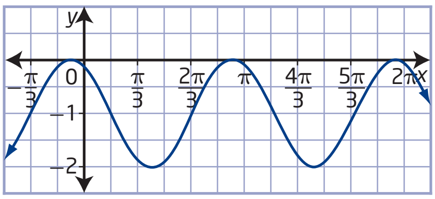
d) $y = \sin \left(\frac{1}{2}\left(\theta - \frac{\pi}{4}\right)\right) + 1$

16. Write an equation for each graph in the form $y = a \sin b(x - c) + d$ and in the form $y = a \cos b(x - c) + d$.

a)



b)



17. Write the equation of the cosine function with amplitude 4, period 300°, phase shift of 30° to the left, and vertical displacement 3 units down.

18. **a)** Graph $y = \tan \theta$ for $-2\pi \le \theta \le 0$.

b) State the equations of the asymptotes for the graph in part a).

19. A Ferris wheel has a radius of 25 m. The centre of the wheel is 26 m above the ground. The wheel rotates twice in 22 min.

a) Determine the equation of the sinusoidal function, $h(x)$, that models the height of a passenger on the ride as a function of time.

b) If a passenger gets on at the bottom of the wheel, when is the passenger 30 m above the ground? Express the answer to the nearest tenth of a minute.

Chapter 6 Trigonometric Identities

20. Determine the non-permissible values for each expression. Then, simplify the expression.

a) $\frac{1 - \cos^2 \theta}{\cos^2 \theta}$ **b)** $\sec x \csc x \tan x$

21. Use a sum or difference identity to determine the exact values of each trigonometric expression.

a) $\sin 195°$ **b)** $\cos \left(-\frac{5\pi}{12}\right)$

22. Write each expression as a single trigonometric ratio and then evaluate.

a) $2 \cos^2 \frac{3\pi}{8} - 1$

b) $\sin 10° \cos 80° + \cos 10° \sin 80°$

c) $\dfrac{\tan \frac{5\pi}{12} + \tan \frac{23\pi}{12}}{1 - \tan \frac{5\pi}{12} \tan \frac{23\pi}{12}}$

23. **a)** Verify the equation $\sin^2 A + \cos^2 A + \tan^2 A = \sec^2 A$ for $A = 30°$.

b) Prove that the equation in part a) is an identity.

24. Consider the equation $\frac{1 + \tan x}{\sec x} = \sin x + \cos x$.

a) Verify graphically that the equation could be an identity.

b) Prove that the equation is an identity for all permissible values of x.

25. Prove the identity $\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \frac{\cos 2\theta}{1 + \sin 2\theta}$ algebraically.

26. Solve each equation. Give the general solution in radians.

a) $\sec^2 x = 4 \tan^2 x$

b) $\sin 2x + \cos x = 0$

27. **a)** Solve $(\sin \theta + \cos \theta)^2 - \sin 2\theta = 1$. Give the general solution in degrees.

b) Is the equation $(\sin \theta + \cos \theta)^2 - \sin 2\theta = 1$ an identity? Explain.

Unit 2 Test

Multiple Choice

For #1 to #8, choose the best answer.

1. If $\tan \theta = \frac{3}{2}$ and $\cos \theta < 0$, then the value of $\cos 2\theta$ is

A $\frac{1}{13}$ **B** $-\frac{5}{13}$

C $\frac{5}{13}$ **D** 1

2. If the point $(3, -5)$ lies on the terminal arm of an angle θ in standard position, the value of $\sin (\pi - \theta)$ is

A $\frac{3}{\sqrt{34}}$ **B** $-\frac{3}{\sqrt{34}}$

C $\frac{5}{\sqrt{34}}$ **D** $-\frac{5}{\sqrt{34}}$

3. The function $y = a \sin b(x - c) + d$ has a range of $\{y \mid -2 \le y \le 6, y \in \mathrm{R}\}$. What are the values of the parameters a and d?

A $a = -2$ and $d = 8$

B $a = 2$ and $d = 4$

C $a = 4$ and $d = 2$

D $a = 8$ and $d = -2$

4. What are the period and phase shift for the function $f(x) = 3 \cos \left(4x + \frac{\pi}{2}\right)$?

A period $= \frac{\pi}{2}$, phase shift $= \frac{\pi}{2}$ units to the left

B period $= 4$, phase shift $= \frac{\pi}{8}$ units to the left

C period $= \frac{\pi}{2}$, phase shift $= \frac{\pi}{8}$ units to the left

D period $= 4$, phase shift $= \frac{\pi}{2}$ units to the left

5. Which sinusoidal function has a graph equivalent to the graph of $y = 3 \sin x$?

A $y = 3 \cos \left(x + \frac{\pi}{2}\right)$

B $y = 3 \cos \left(x - \frac{\pi}{2}\right)$

C $y = 3 \cos \left(x - \frac{\pi}{4}\right)$

D $y = 3 \cos \left(x + \frac{\pi}{4}\right)$

6. The function $y = \tan x$, where x is in degrees, is

A defined for all values of x

B undefined when $x = \pm 1°$

C undefined when $x = 180°n, n \in \mathrm{I}$

D undefined when $x = 90° + 180°n, n \in \mathrm{I}$

7. The expression $\frac{\sin \theta + \tan \theta}{1 + \cos \theta}$ is equivalent to

A $\sin \theta$

B $\cos \theta$

C $\tan \theta$

D $\cot \theta$

8. Which of the following is not an identity?

A $\frac{\sec \theta \csc \theta}{\cot \theta} = \sec \theta$

B $\tan^2 \theta - \sin^2 \theta = \sin^2 \theta \tan^2 \theta$

C $\frac{1 - \cos 2\theta}{2} = \sin^2 \theta$

D $\frac{\tan^2 \theta}{1 + \tan^2 \theta} = \sin^2 \theta$

Numerical Response

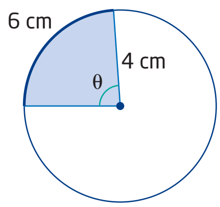
Copy and complete the statements in #9 to #13.

9. The exact value of $\sin \frac{17\pi}{3}$ is ■.

10. Point $\mathrm{P}\left(x, \frac{\sqrt{5}}{3}\right)$ is on the unit circle. The possible values of x are ■ and ■.

11. If $\cos \theta = \frac{-5}{13}$ and $\frac{\pi}{2} \le \theta \le \pi$, then the exact value of $\sin \left(\theta + \frac{\pi}{4}\right)$ is ■.

12. An arc of a circle subtends a central angle θ. The length of the arc is 6 cm and the radius is 4 cm. The measures of θ in radians and in degrees, to the nearest tenth of a unit, are ■ and ■.



13. The solutions to $\sqrt{3}\sec\theta - 2 = 0$ for $-2\pi \le \theta \le 2\pi$, as exact values, are ■, ■, ■, and ■.

Written Response

14. Consider the angle $\theta = -\frac{5\pi}{3}$.
   a) Draw the angle in standard position.
   b) Convert the angle measure to degrees.
   c) Write an expression for all angles that are coterminal with θ, in radians.
   d) Is $\frac{10\pi}{3}$ coterminal with θ? Justify your answer.

15. Solve $5\sin^2\theta + 3\sin\theta - 2 = 0$, $0 \le \theta \le 2\pi$, algebraically. Give your answers to the nearest thousandth of a radian.

16. Pat solved the equation $4\sin^2 x = 3$, $0 \le x \le 2\pi$, as follows:

$$4\sin^2 x = 3$$
$$2\sin x = \sqrt{3}$$
$$\sin x = \frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3}, \frac{2\pi}{3}$$

Sam checks the answer graphically and says that there are four zeros in the given domain. Who is correct? Identify the error that the other person made.

17. a) Sketch the graph of the function $f(x) = 3\sin\frac{1}{2}(x + 60°) - 1$ for $-360° \le x \le 360°$.
   b) State the range of the function.
   c) Identify the amplitude, period, phase shift, and vertical displacement for the function.
   d) Determine the roots of the equation $3\sin\frac{1}{2}(x + 60°) - 1 = 0$. Give your answers to the nearest degree.

18. a) Use technology to graph $f(\theta) = 2\cot\theta\sin^2\theta$ over the domain $0 \le \theta \le 2\pi$.
   b) Determine an equation equivalent to $f(\theta)$ in the form $g(\theta) = a\sin[b(\theta - c)] + d$, where a, b, c, and d are constants.
   c) Prove algebraically that $f(\theta) = g(\theta)$.

19. Consider the equation $\tan x + \frac{1}{\tan x} = \frac{\sec x}{\sin x}$.
   a) Verify that the equation is true for $x = \frac{2\pi}{3}$.
   b) What are the non-permissible values for the equation?
   c) Prove that the equation is an identity for all permissible values of x.

20. The predicted height, h, in metres, of the tides at Prince Rupert, British Columbia, for one day in February is approximated by the function $h(t) = 2.962\sin(0.508t - 0.107) + 3.876$, where t is the time, in hours, since midnight.
   a) Predict the maximum height of the tide on this day.
   b) Determine the period of the sinusoidal function.
   c) Predict the height of the tide at 12 noon on this day.

Web Link

To search for data on predicted heights and times of tides at various locations across Canada, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

Marina at Prince Rupert