Unit 3

Exponential and Logarithmic Functions

Exponential and logarithmic functions can be used to describe and solve a wide range of problems. Some of the questions that can be answered using these two types of functions include:

- How much will your bank deposit be worth in five years, if it is compounded monthly?
- How will your car loan payment change if you pay it off in three years instead of four?
- How acidic is a water sample with a pH of 8.2?
- How long will a medication stay in your bloodstream with a concentration that allows it to be effective?
- How thick should the walls of a spacecraft be in order to protect the crew from harmful radiation?

In this unit, you will explore a variety of situations that can modelled with an exponential function or its inverse, the logarithmic function. You will learn techniques for solving various problems, such as those posed above.

Looking Ahead

In this unit, you will solve problems involving…

- exponential functions and equations
- logarithmic functions and equations

Unit 3 Project **At the Movies**

In 2010, Canadian and American movie-goers spent $10.6 billion on tickets, or 33% of the worldwide box office ticket sales. Of the films released in 2010, only 25 were in 3D, but they brought in $2.2 billion of the ticket sales!

You will examine box office revenues for newly released movies, investigate graphs of the revenue over time, determine the function that best represents the data and graph, and use this function to make predictions.

In this project, you will explore the use of mathematics to model box office revenues for a movie of your choice.

CHAPTER 7

Exponential Functions

In the 1920s, watch companies produced glow-in-the-dark dials by using radioluminescent paint, which was made of zinc sulphide mixed with radioactive radium salts. Today, a material called tritium is used in wristwatches and other equipment such as aircraft instruments. In commercial use, the tritium gas is put into tiny vials of borosilicate glass that are placed on the hands and hour markers of a watch dial.

Both radium and tritium are radioactive materials that decay into other elements by emitting different types of radiation. The rate at which radioactive materials decay can be modelled using exponential functions. Exponential functions can also be used to model situations where there is geometric growth, such as in bacterial colonies and financial investments.

In this chapter, you will study exponential functions and use them to solve a variety of problems.

Did You Know?

Radium was once an additive in toothpaste, hair creams, and even food items due to its supposed curative powers. Once it was discovered that radium is over one million times as radioactive as the same mass of uranium, these products were prohibited because of their serious adverse health effects.

Key Terms

exponential function
exponential growth
exponential decay
half-life
exponential equation

Career Link

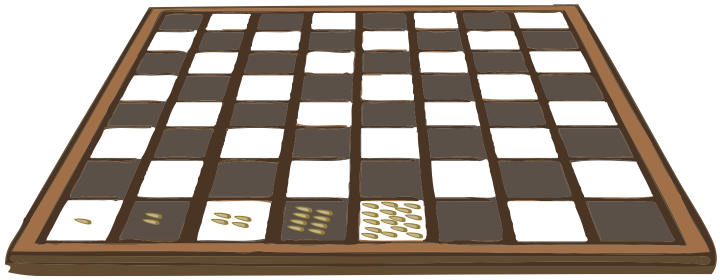
Chemistry helps us understand the world around us and allows us to create new substances and materials. Chemists synthesize, discover, and characterize molecules and their chemical reactions. They may develop products such as synthetic fibres and pharmaceuticals, or processes such as sustainable solutions for energy production and greener methods of manufacturing materials.

Web Link

To learn more about a career in chemistry, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

7.1

Characteristics of Exponential Functions



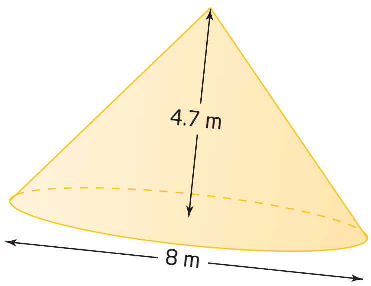
Focus on...

- analysing graphs of exponential functions
- solving problems that involve exponential growth or decay

The following ancient fable from India has several variations, but each makes the same point.

> *When the creator of the game of chess showed his invention to the ruler of the country, the ruler was so pleased that he gave the inventor the right to name his prize for the invention. The man, who was very wise, asked the king that he be given one grain of rice for the first square of the chessboard, two for the second one, four for the third one, and so on. The ruler quickly accepted the inventor's offer, believing the man had made a mistake in not asking for more.*
>
> *By the time he was compensated for half the chessboard, the man owned all of the rice in the country, and, by the sixty-fourth square, the ruler owed him almost 20 000 000 000 000 000 000 grains of rice.*

Amount of Rice on 32nd Square



The final amount of rice is approximately equal to the volume of a mountain 140 times as tall as Mount Everest. This is an example of how things grow exponentially. What exponential function can be used to model this situation?

Investigate Characteristics of Exponential Functions

Materials

- graphing technology

Explore functions of the form $y = c^x$.

1. Consider the function $y = 2^x$.

a) Graph the function.

b) Describe the shape of the graph.

2. Determine the following and justify your reasoning.

a) the domain and the range of the function $y = 2^x$

b) the y-intercept

c) the x-intercept

d) the equation of the horizontal line (asymptote) that the graph approaches as the values of x get very small

3. Select at least two different values for c in $y = c^x$ that are greater than 2. Graph your functions. Compare each graph to the graph of $y = 2^x$. Describe how the graphs are similar and how they are different.
4. Select at least two different values for c in $y = c^x$ that are between 0 and 1. Graph your functions. How have the graphs changed compared to those from steps 2 and 3?
5. Predict what the graph will look like if $c < 0$. Confirm your prediction using a table of values and a graph.

Reflect and Respond

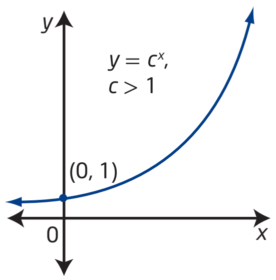
6. a) Summarize how the value of c affects the shape and characteristics of the graph of $y = c^x$.
   b) Predict what will happen when $c = 1$. Explain.

Link the Ideas

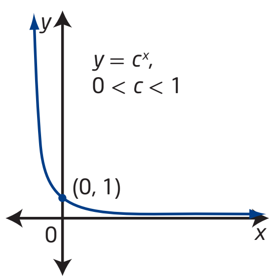
The graph of an **exponential function**, such as $y = c^x$, is increasing for $c > 1$, decreasing for $0 < c < 1$, and neither increasing nor decreasing for $c = 1$. From the graph, you can determine characteristics such as domain and range, any intercepts, and any asymptotes.

exponential function

- a function of the form $y = c^x$, where c is a constant ($c > 0$) and x is a variable



Increasing



Decreasing

Why is the definition of an exponential function restricted to positive values of c?

Did You Know?

Any letter can be used to represent the base in an exponential function. Some other common forms are $y = a^x$ and $y = b^x$. In this chapter, you will use the letter c. This is to avoid any confusion with the transformation parameters, a, b, h, and k, that you will apply in Section 7.2.

Example 1

Analyse the Graph of an Exponential Function

Graph each exponential function. Then identify the following:
- the domain and range
- the x-intercept and y-intercept, if they exist
- whether the graph represents an increasing or a decreasing function
- the equation of the horizontal asymptote

a) $y = 4^x$

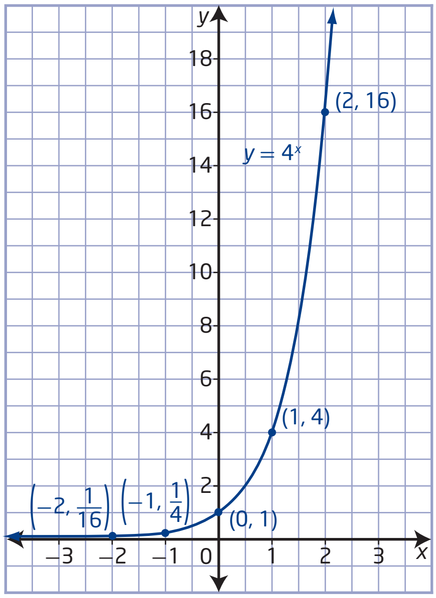
b) $f(x) = \left(\frac{1}{2}\right)^x$

Solution

a) **Method 1: Use Paper and Pencil**

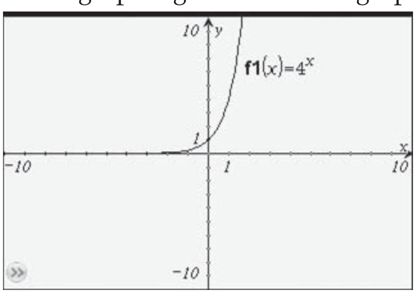
Use a table of values to graph the function.
Select integral values of x that make it easy to calculate the corresponding values of y for $y = 4^x$.

x	y
−2	$\frac{1}{16}$
−1	$\frac{1}{4}$
0	1
1	4
2	16



Method 2: Use a Graphing Calculator

Use a graphing calculator to graph $y = 4^x$.



The function is defined for all values of x. Therefore, the domain is $\{x \mid x \in \mathrm{R}\}$.

The function has only positive values for y. Therefore, the range is $\{y \mid y > 0, y \in \mathrm{R}\}$.

The graph never intersects the x-axis, so there is no x-intercept.

The graph crosses the y-axis at $y = 1$, so the y-intercept is 1.

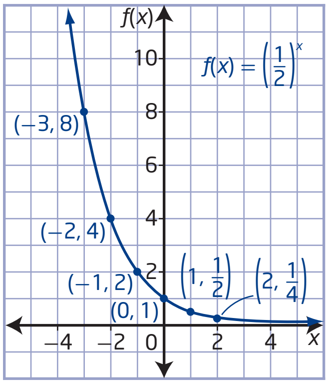
The graph rises to the right throughout its domain, indicating that the values of y increase as the values of x increase. Therefore, the function is increasing over its domain.

Since the graph approaches the line $y = 0$ as the values of x get very small, $y = 0$ is the equation of the horizontal asymptote.

b) **Method 1: Use Paper and Pencil**

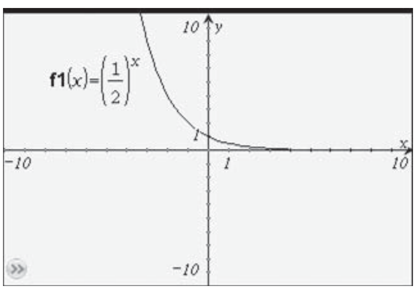
Use a table of values to graph the function.
Select integral values of x that make it easy to calculate the corresponding values of y for $f(x) = \left(\frac{1}{2}\right)^x$.

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$



Method 2: Use a Graphing Calculator

Use a graphing calculator to graph $f(x) = \left(\frac{1}{2}\right)^x$.



The function is defined for all values of x. Therefore, the domain is $\{x \mid x \in \mathrm{R}\}$.

The function has only positive values for y. Therefore, the range is $\{y \mid y > 0, y \in \mathrm{R}\}$.

The graph never intersects the x-axis, so there is no x-intercept.

The graph crosses the y-axis at $y = 1$, so the y-intercept is 1.

Why do the graphs of these exponential functions have a y-intercept of 1?

The graph falls to the right throughout its domain, indicating that the values of y decrease as the values of x increase. Therefore, the function is decreasing over its domain.

Since the graph approaches the line $y = 0$ as the values of x get very large, $y = 0$ is the equation of the horizontal asymptote.

Your Turn

Graph the exponential function $y = 3^x$ without technology. Identify the following:

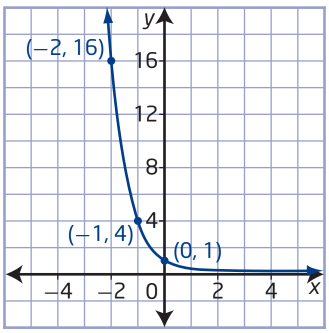
- the domain and range
- the x-intercept and the y-intercept, if they exist
- whether the graph represents an increasing or a decreasing function
- the equation of the horizontal asymptote

Verify your results using graphing technology.

Example 2

Write the Exponential Function Given Its Graph

What function of the form $y = c^x$ can be used to describe the graph shown?



Solution

Look for a pattern in the ordered pairs from the graph.

x	y
–2	16
–1	4
0	1

As the value of x increases by 1 unit, the value of y decreases by a factor of $\frac{1}{4}$. Therefore, for this function, $c = \frac{1}{4}$.

Choose a point other than (0, 1) to substitute into the function $y = \left(\frac{1}{4}\right)^x$ to verify that the function is correct. Try the point (–2, 16).

Why should you not use the point (0, 1) to verify that the function is correct?

Check:

Left Side	Right Side
y	$\left(\frac{1}{4}\right)^x$
$= 16$	$= \left(\frac{1}{4}\right)^{-2}$
	$= \dfrac{1}{\left(\frac{1}{4}\right)^2}$
	$= \left(\frac{4}{1}\right)^2$
	$= 16$

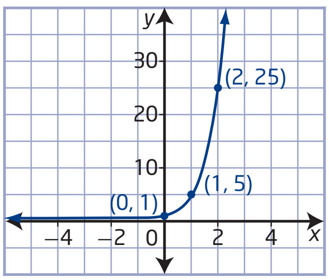
Why is the power with a negative exponent, $\left(\frac{1}{4}\right)^{-2}$, equivalent to the reciprocal of the power with a positive exponent, $\dfrac{1}{\left(\frac{1}{4}\right)^2}$?

The right side equals the left side, so the function that describes the graph is $y = \left(\frac{1}{4}\right)^x$.

What is another way of expressing this exponential function?

Your Turn

What function of the form $y = c^x$ can be used to describe the graph shown?



Exponential functions of the form $y = c^x$, where $c > 1$, can be used to model **exponential growth**. Exponential functions with $0 < c < 1$ can be used to model **exponential decay**.

exponential growth

- an increasing pattern of values that can be modelled by a function of the form $y = c^x$, where $c > 1$

exponential decay

- a decreasing pattern of values that can be modelled by a function of the form $y = c^x$, where $0 < c < 1$

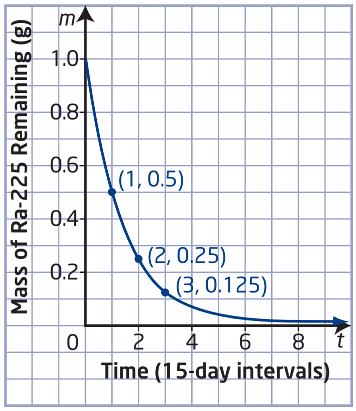
half-life

- the length of time for an unstable element to spontaneously decay to one half its original mass

Example 3

Application of an Exponential Function

A radioactive sample of radium (Ra-225) has a **half-life** of 15 days. The mass, m, in grams, of Ra-225 remaining over time, t, in 15-day intervals, can be modelled using the exponential graph shown.



a) What is the initial mass of Ra-225 in the sample? What value does the mass of Ra-225 remaining approach as time passes?

b) What are the domain and range of this function?

c) Write the exponential decay model that relates the mass of Ra-225 remaining to time, in 15-day intervals.

d) Estimate how many days it would take for Ra-225 to decay to $\frac{1}{30}$ of its original mass.

Solution

a) From the graph, the m-intercept is 1. So, the initial mass of Ra-225 in the sample is 1 g.

The graph is decreasing by a constant factor over time, representing exponential decay. It appears to approach $m = 0$ or 0 g of Ra-225 remaining in the sample.

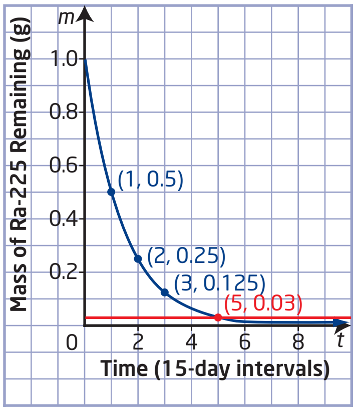
b) From the graph, the domain of the function is $\{t \mid t \geq 0, t \in \mathrm{R}\}$, and the range of the function is $\{m \mid 0 < m \leq 1, m \in \mathrm{R}\}$.

c) The exponential decay model that relates the mass of Ra-225 remaining to time, in 15-day intervals, is the function $m(t) = \left(\frac{1}{2}\right)^t$.

Why is the base of the exponential function $\frac{1}{2}$?

d) Method 1: Use the Graph of the Function

$\frac{1}{30}$ of 1 g is equivalent to 0.0333… or approximately 0.03 g. Locate this approximate value on the vertical axis of the graph and draw a horizontal line until it intersects the graph of the exponential function.



The horizontal line appears to intersect the graph at the point (5, 0.03). Therefore, it takes approximately five 15-day intervals, or 75 days, for Ra-225 to decay to $\frac{1}{30}$ of its original mass.

Method 2: Use a Table of Values

$\frac{1}{30}$ of 1 g is equivalent to $\frac{1}{30}$ g or approximately 0.0333 g.

Create a table of values for $m(t) = \left(\frac{1}{2}\right)^t$.

t	m
1	0.5
2	0.25
3	0.125
4	0.0625
5	0.031 25
6	0.015 625

The table shows that the number of 15-day intervals is between 4 and 5. You can determine a better estimate by looking at values between these numbers.

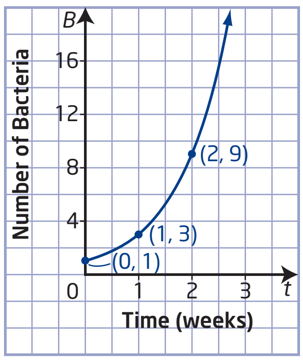
Since 0.0333 is much closer to 0.031 25, try 4.8: $\left(\frac{1}{2}\right)^{4.8} \approx 0.0359$. This is greater than 0.0333.

Try 4.9: $\left(\frac{1}{2}\right)^{4.9} \approx 0.0335$.

Therefore, it will take approximately 4.9 15-day intervals, or 73.5 days, for Ra-225 to decay to $\frac{1}{30}$ of its original mass.

Your Turn

Under ideal circumstances, a certain bacteria population triples every week. This is modelled by the following exponential graph.



a) What are the domain and range of this function?

b) Write the exponential growth model that relates the number, B, of bacteria to the time, t, in weeks.

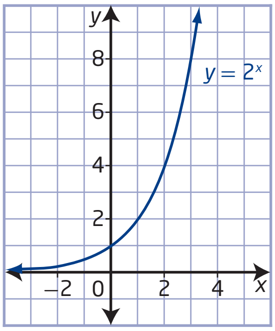
c) Determine approximately how many days it would take for the number of bacteria to increase to eight times the quantity on day 1.

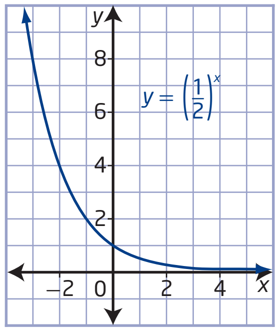
Did You Know?

Exponential functions can be used to model situations involving continuous or discrete data. For example, problems involving radioactive decay are based on continuous data, while those involving populations are based on discrete data. In the case of discrete data, the continuous model will only be valid for a restricted domain.

Key Ideas

- An exponential function of the form $y = c^x$, $c > 0$,
  - is increasing for $c > 1$
  - is decreasing for $0 < c < 1$
  - is neither increasing nor decreasing for $c = 1$
  - has a domain of $\{x \mid x \in \mathrm{R}\}$
  - has a range of $\{y \mid y > 0, y \in \mathrm{R}\}$
  - has a y-intercept of 1
  - has no x-intercept
  - has a horizontal asymptote at $y = 0$





Check Your Understanding

Practise

1. Decide whether each of the following functions is exponential. Explain how you can tell.

a) $y = x^3$

b) $y = 6^x$

c) $y = x^{\frac{1}{2}}$

d) $y = 0.75^x$

2. Consider the following exponential functions:

- $f(x) = 4^x$
- $g(x) = \left(\frac{1}{4}\right)^x$
- $h(x) = 2^x$

a) Which is greatest when $x = 5$?

b) Which is greatest when $x = -5$?

c) For which value of x do all three functions have the same value? What is this value?

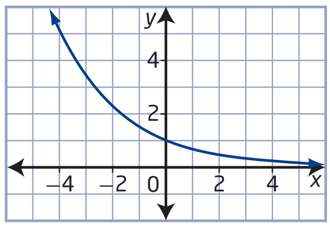
3. Match each exponential function to its corresponding graph.

a) $y = 5^x$

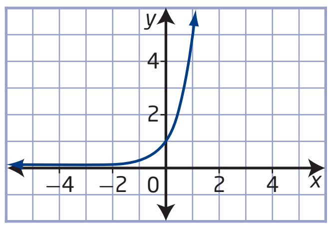
b) $y = \left(\frac{1}{4}\right)^x$

c) $y = \left(\frac{2}{3}\right)^x$

A



B



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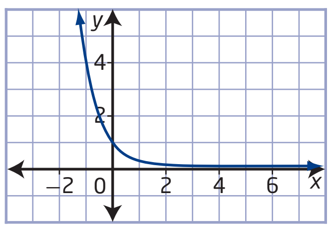
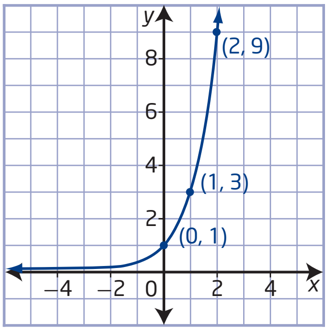
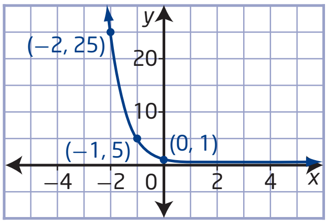


4. Write the function equation for each graph of an exponential function.

a)



b)



5. Sketch the graph of each exponential function. Identify the domain and range, the y-intercept, whether the function is increasing or decreasing, and the equation of the horizontal asymptote.

a) $g(x) = 6^x$

b) $h(x) = 3.2^x$

c) $f(x) = \left(\frac{1}{10}\right)^x$

d) $k(x) = \left(\frac{3}{4}\right)^x$

Apply

6. Each of the following situations can be modelled using an exponential function. Indicate which situations require a value of $c > 1$ (growth) and which require a value of $0 < c < 1$ (decay). Explain your choices.

a) Bacteria in a Petri dish double their number every hour.

b) The half-life of the radioactive isotope actinium-225 is 10 days.

c) As light passes through every 1-m depth of water in a pond, the amount of light available decreases by 20%.

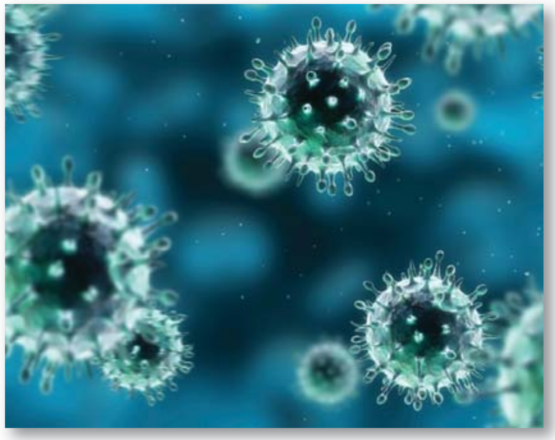
d) The population of an insect colony triples every day.

7. A flu virus is spreading through the student population of a school according to the function $N = 2^t$, where N is the number of people infected and t is the time, in days.

a) Graph the function. Explain why the function is exponential.

b) How many people have the virus at each time?

i) at the start when $t = 0$
ii) after 1 day
iii) after 4 days
iv) after 10 days



8. If a given population has a constant growth rate over time and is never limited by food or disease, it exhibits exponential growth. In this situation, the growth rate alone controls how quickly (or slowly) the population grows. If a population, P, of fish, in hundreds, experiences exponential growth at a rate of 10% per year, it can be modelled by the exponential function $P(t) = 1.1^t$, where t is time, in years.

a) Why is the base for the exponential function that models this situation 1.1?

b) Graph the function $P(t) = 1.1^t$. What are the domain and range of the function?

c) If the same population of fish decreased at a rate of 5% per year, how would the base of the exponential model change?

d) Graph the new function from part c). What are the domain and range of this function?

9. Scuba divers know that the deeper they dive, the more light is absorbed by the water above them. On a dive, Petra's light meter shows that the amount of light available decreases by 10% for every 10 m that she descends.

a) Write the exponential function that relates the amount, L, as a percent expressed as a decimal, of light available to the depth, d, in 10-m increments.

b) Graph the function.

c) What are the domain and range of the function for this situation?

d) What percent of light will reach Petra if she dives to a depth of 25 m?

10. The CANDU (CANada Deuterium Uranium) reactor is a Canadian-invented pressurized heavy-water reactor that uses uranium-235 (U-235) fuel with a half-life of approximately 700 million years.

a) What exponential function can be used to represent the radioactive decay of 1 kg of U-235? Define the variables you use.

b) Graph the function.

c) How long will it take for 1 kg of U-235 to decay to 0.125 kg?

d) Will the sample in part c) decay to 0 kg? Explain.

Did You Know?

Canada is one of the world's leading uranium producers, accounting for 18% of world primary production. All of the uranium produced in Canada comes from Saskatchewan mines. The energy potential of Saskatchewan's uranium reserves is approximately equivalent to 4.5 billion tonnes of coal or 17.5 billion barrels of oil.

11. Money in a savings account earns compound interest at a rate of 1.75% per year. The amount, A, of money in an account can be modelled by the exponential function $A = P(1.0175)^n$, where P is the amount of money first deposited into the savings account and n is the number of years the money remains in the account.

a) Graph this function using a value of $P = \$1$ as the initial deposit.

b) Approximately how long will it take for the deposit to triple in value?

c) Does the amount of time it takes for a deposit to triple depend on the value of the initial deposit? Explain.

d) In finance, the *rule of 72* is a method of estimating an investment's doubling time when interest is compounded annually. The number 72 is divided by the annual interest rate to obtain the approximate number of years required for doubling. Use your graph and the rule of 72 to approximate the doubling time for this investment.

12. Statistics indicate that the world population since 1995 has been growing at a rate of about 1.27% per year. United Nations records estimate that the world population in 2011 was approximately 7 billion. Assuming the same exponential growth rate, when will the population of the world be 9 billion?

Extend

13. a) On the same set of axes, sketch the graph of the function $y = 5^x$, and then sketch the graph of the inverse of the function by reflecting its graph in the line $y = x$.

b) How do the characteristics of the graph of the inverse of the function relate to the characteristics of the graph of the original exponential function?

c) Express the equation of the inverse of the exponential function in terms of y. That is, write $x = F(y)$.

14. The Krumbein phi scale is used in geology to classify sediments such as silt, sand, and gravel by particle size. The scale is modelled by the function $D(\varphi) = 2^{-\varphi}$, where D is the diameter of the particle, in millimetres, and φ is the Krumbein scale value. Fine sand has a Krumbein scale value of approximately 3. Coarse gravel has a Krumbein scale value of approximately -5.

A sampler showing grains sorted from silt to very coarse sand.

a) Why would a coarse material have a negative scale value?

b) How does the diameter of fine sand compare with the diameter of coarse gravel?

15. Typically, compound interest for a savings account is calculated every month and deposited into the account at that time. The interest could also be calculated daily, or hourly, or even by the second. When the period of time is infinitesimally small, the interest calculation is called continuous compounding. The exponential function that models this situation is $A(t) = Pe^{rt}$, where P is the amount of the initial deposit, r is the annual rate of interest as a decimal value, t is the number of years, and e is the base (approximately equal to 2.7183).

a) Use graphing technology to estimate the doubling period, assuming an annual interest rate of 2% and continuous compounding.

b) Use graphing technology to estimate the doubling period using the compound interest formula $A = P(1 + i)^n$.

c) How do the results in parts a) and b) compare? Which method results in a shorter doubling period?

Did You Know?

The number *e* is irrational because it cannot be expressed as a ratio of integers. It is sometimes called *Euler's number* after the Swiss mathematician Leonhard Euler (pronounced "oiler").

Create Connections

C1 Consider the functions $f(x) = 3x$, $g(x) = x^3$, and $h(x) = 3^x$.

a) Graph each function.

b) List the key features for each function: domain and range, intercepts, and equations of any asymptotes.

c) Identify key features that are common to each function.

d) Identify key features that are different for each function.

C2 Consider the function $f(x) = (-2)^x$.

a) Copy and complete the table of values.

x	$f(x)$
0	
1	
2	
3	
4	
5	

b) Plot the ordered pairs.

c) Do the points form a smooth curve? Explain.

d) Use technology to try to evaluate $f\left(\frac{1}{2}\right)$ and $f\left(\frac{5}{2}\right)$. Use numerical reasoning to explain why these values are undefined.

e) Use these results to explain why exponential functions are defined to only include positive bases.

7.2

Transformations of Exponential Functions

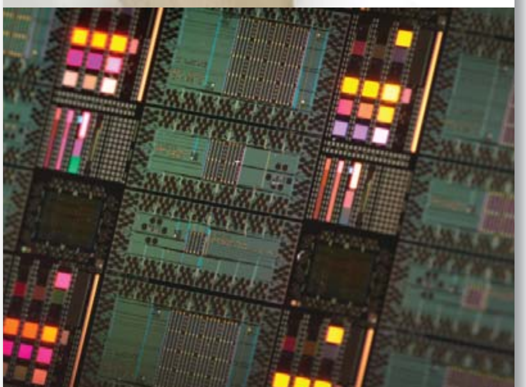
Focus on...

- applying translations, stretches, and reflections to the graphs of exponential functions
- representing these transformations in the equations of exponential functions
- solving problems that involve exponential growth or decay

Transformations of exponential functions are used to model situations such as population growth, carbon dating of samples found at archaeological digs, the physics of nuclear chain reactions, and the processing power of computers.

In this section, you will examine transformations of exponential functions and the impact the transformations have on the corresponding graph.


Taking a sample for carbon dating


D-wave wafer processor

Did You Know?

Moore's law describes a trend in the history of computing hardware. It states that the number of transistors that can be placed on an integrated circuit will double approximately every 2 years. The trend has continued for over half a century.

Investigate Transforming an Exponential Function

Materials

- graphing technology

Apply your prior knowledge of transformations to predict the effects of translations, stretches, and reflections on exponential functions of the form $f(x) = a(c)^{b(x-h)} + k$ and their associated graphs.

A: The Effects of Parameters *h* and *k* on the Function $f(x) = a(c)^{b(x-h)} + k$

1. a) Graph each set of functions on one set of coordinate axes. Sketch the graphs in your notebook.

Set A	Set B
i) $f(x) = 3^x$	**i)** $f(x) = 2^x$
ii) $f(x) = 3^x + 2$	**ii)** $f(x) = 2^{x-3}$
iii) $f(x) = 3^x - 4$	**iii)** $f(x) = 2^{x+1}$

b) Compare the graphs in set A. For any constant k, describe the relationship between the graphs of $f(x) = 3^x$ and $f(x) = 3^x + k$.

c) Compare the graphs in set B. For any constant h, describe the relationship between the graphs of $f(x) = 2^x$ and $f(x) = 2^{x-h}$.

Reflect and Respond

2. Describe the roles of the parameters h and k in functions of the form $f(x) = a(c)^{b(x-h)} + k$.

B: The Effects of Parameters *a* and *b* on the Function $f(x) = a(c)^{b(x-h)} + k$

3. a) Graph each set of functions on one set of coordinate axes. Sketch the graphs in your notebook.

Set C	Set D
i) $f(x) = \left(\frac{1}{2}\right)^x$	**i)** $f(x) = 2^x$
ii) $f(x) = 3\left(\frac{1}{2}\right)^x$	**ii)** $f(x) = 2^{3x}$
iii) $f(x) = \frac{3}{4}\left(\frac{1}{2}\right)^x$	**iii)** $f(x) = 2^{\frac{1}{3}x}$
iv) $f(x) = -4\left(\frac{1}{2}\right)^x$	**iv)** $f(x) = 2^{-2x}$
v) $f(x) = -\frac{1}{3}\left(\frac{1}{2}\right)^x$	**v)** $f(x) = 2^{-\frac{2}{3}x}$

What effect does the negative sign have on the graph?

b) Compare the graphs in set C. For any real value a, describe the relationship between the graphs of $f(x) = \left(\frac{1}{2}\right)^x$ and $f(x) = a\left(\frac{1}{2}\right)^x$.

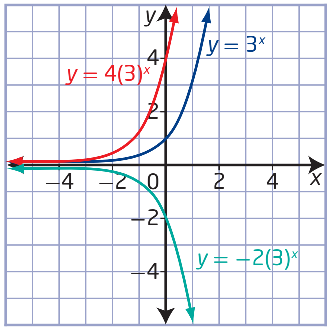
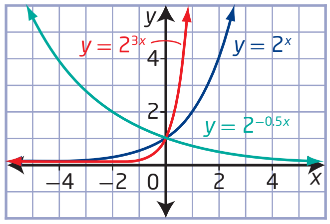
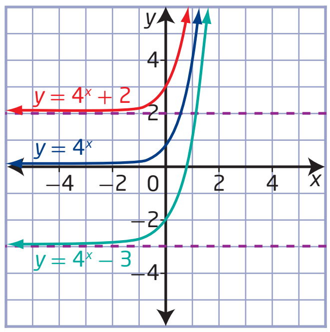
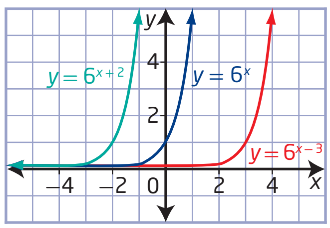
c) Compare the graphs in set D. For any real value b, describe the relationship between the graphs of $f(x) = 2^x$ and $f(x) = 2^{bx}$.

Reflect and Respond

4. Describe the roles of the parameters a and b in functions of the form $f(x) = a(c)^{b(x-h)} + k$.

Link the Ideas

The graph of a function of the form $f(x) = a(c)^{b(x-h)} + k$ is obtained by applying transformations to the graph of the base function $y = c^x$, where $c > 0$.

Parameter	Transformation	Example
a	• Vertical stretch about the x-axis by a factor of $\lvert a \rvert$ • For $a < 0$, reflection in the x-axis • $(x, y) \rightarrow (x, ay)$	
b	• Horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$ • For $b < 0$, reflection in the y-axis • $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$	
k	• Vertical translation up or down • $(x, y) \rightarrow (x, y + k)$	
h	• Horizontal translation left or right • $(x, y) \rightarrow (x + h, y)$	

An accurate sketch of a transformed graph is obtained by applying the transformations represented by a and b before the transformations represented by h and k.

How does this compare to your past experience with transformations?

Example 1

Apply Transformations to Sketch a Graph

Consider the base function $y = 3^x$. For each transformed function,

i) state the parameters and describe the corresponding transformations

ii) create a table to show what happens to the given points under each transformation

$y = 3^x$
$\left(-1, \frac{1}{3}\right)$
(0, 1)
(1, 3)
(2, 9)
(3, 27)

iii) sketch the graph of the base function and the transformed function

iv) describe the effects on the domain, range, equation of the horizontal asymptote, and intercepts

a) $y = 2(3)^{x-4}$

b) $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$

Solution

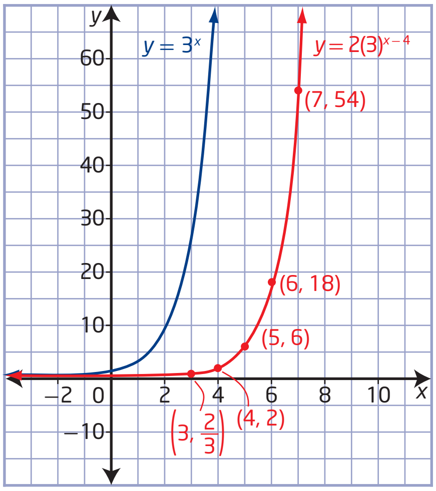
a) **i)** Compare the function $y = 2(3)^{x-4}$ to $y = a(c)^{b(x-h)} + k$ to determine the values of the parameters.

- $b = 1$ corresponds to no horizontal stretch.
- $a = 2$ corresponds to a vertical stretch of factor 2. Multiply the y-coordinates of the points in column 1 by 2.
- $h = 4$ corresponds to a translation of 4 units to the right. Add 4 to the x-coordinates of the points in column 2.
- $k = 0$ corresponds to no vertical translation.

ii) Add columns to the table representing the transformations.

$y = 3^x$	$y = 2(3)^x$	$y = 2(3)^{x-4}$
$\left(-1, \frac{1}{3}\right)$	$\left(-1, \frac{2}{3}\right)$	$\left(3, \frac{2}{3}\right)$
(0, 1)	(0, 2)	(4, 2)
(1, 3)	(1, 6)	(5, 6)
(2, 9)	(2, 18)	(6, 18)
(3, 27)	(3, 54)	(7, 54)

iii) To sketch the graph, plot the points from column 3 and draw a smooth curve through them.



iv) The domain remains the same: $\{x \mid x \in \mathrm{R}\}$.

The range also remains unchanged: $\{y \mid y > 0, y \in \mathrm{R}\}$.

The equation of the asymptote remains as $y = 0$.

There is still no x-intercept, but the y-intercept changes to $\frac{2}{81}$ or approximately 0.025.

b) i) Compare the function $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$ to $y = a(c)^{b(x-h)} + k$ to determine the values of the parameters.

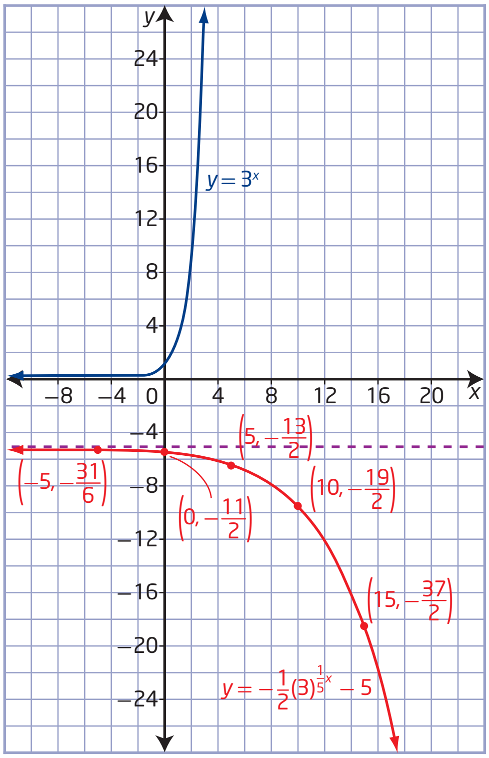
- $b = \frac{1}{5}$ corresponds to a horizontal stretch of factor 5. Multiply the x-coordinates of the points in column 1 by 5.
- $a = -\frac{1}{2}$ corresponds to a vertical stretch of factor $\frac{1}{2}$ and a reflection in the x-axis. Multiply the y-coordinates of the points in column 2 by $-\frac{1}{2}$.
- $h = 0$ corresponds to no horizontal translation.
- $k = -5$ corresponds to a translation of 5 units down. Subtract 5 from the y-coordinates of the points in column 3.

ii) Add columns to the table representing the transformations.

$y = 3^x$	$y = 3^{\frac{1}{5}x}$	$y = -\frac{1}{2}(3)^{\frac{1}{5}x}$	$y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$
$\left(-1, \frac{1}{3}\right)$	$\left(-5, \frac{1}{3}\right)$	$\left(-5, -\frac{1}{6}\right)$	$\left(-5, -\frac{31}{6}\right)$
(0, 1)	(0, 1)	$\left(0, -\frac{1}{2}\right)$	$\left(0, -\frac{11}{2}\right)$
(1, 3)	(5, 3)	$\left(5, -\frac{3}{2}\right)$	$\left(5, -\frac{13}{2}\right)$
(2, 9)	(10, 9)	$\left(10, -\frac{9}{2}\right)$	$\left(10, -\frac{19}{2}\right)$
(3, 27)	(15, 27)	$\left(15, -\frac{27}{2}\right)$	$\left(15, -\frac{37}{2}\right)$

iii) To sketch the graph, plot the points from column 4 and draw a smooth curve through them.

Why do the exponential curves have different horizontal asymptotes?



iv) The domain remains the same: $\{x \mid x \in \mathrm{R}\}$.

The range changes to $\{y \mid y < -5, y \in \mathrm{R}\}$ because the graph of the transformed function only exists below the line $y = -5$.

The equation of the asymptote changes to $y = -5$.

There is still no x-intercept, but the y-intercept changes to $-\frac{11}{2}$ or -5.5.

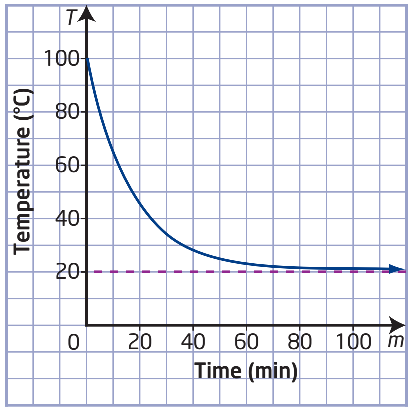
Your Turn

Transform the graph of $y = 4^x$ to sketch the graph of $y = 4^{-2(x + 5)} - 3$. Describe the effects on the domain, range, equation of the horizontal asymptote, and intercepts.

Example 2

Use Transformations of an Exponential Function to Model a Situation

A cup of water is heated to 100 °C and then allowed to cool in a room with an air temperature of 20 °C. The temperature, T, in degrees Celsius, is measured every minute as a function of time, m, in minutes, and these points are plotted on a coordinate grid. It is found that the temperature of the water decreases exponentially at a rate of 25% every 5 min. A smooth curve is drawn through the points, resulting in the graph shown.



a) What is the transformed exponential function in the form $y = a(c)^{b(x-h)} + k$ that can be used to represent this situation?

b) Describe how each of the parameters in the transformed function relates to the information provided.

Solution

a) Since the water temperature decreases by 25% for each 5-min time interval, the base function must be $T(t) = \left(\frac{3}{4}\right)^t$, where T is the temperature and t is the time, in 5-min intervals.

Why is the base of the exponential function $\frac{3}{4}$ when the temperature is reduced by 25%?

The exponent t can be replaced by the rational exponent $\frac{m}{5}$, where m represents the number of minutes: $T(m) = \left(\frac{3}{4}\right)^{\frac{m}{5}}$.

What is the value of the exponent $\frac{m}{5}$ when $m = 5$? How does this relate to the exponent t in the first version of the function?

The asymptote at $T = 20$ means that the function has been translated vertically upward 20 units. This is represented in the function as

$T(m) = \left(\frac{3}{4}\right)^{\frac{m}{5}} + 20$.

The T-intercept of the graph occurs at (0, 100). So, there must be a vertical stretch factor, a. Use the coordinates of the T-intercept to determine a.

$$\begin{aligned} T(m) &= a\left(\frac{3}{4}\right)^{\frac{m}{5}} + 20 \\ 100 &= a\left(\frac{3}{4}\right)^{\frac{0}{5}} + 20 \\ 100 &= a(1) + 20 \\ 80 &= a \end{aligned}$$

Substitute $a = 80$ into the function: $T(m) = 80\left(\frac{3}{4}\right)^{\frac{m}{5}} + 20$.

Check: Substitute $m = 20$ into the function. Compare the result to the graph.

$$T(m) = 80\left(\frac{3}{4}\right)^{\frac{m}{5}} + 20$$
$$T(20) = 80\left(\frac{3}{4}\right)^{\frac{20}{5}} + 20$$
$$= 80\left(\frac{3}{4}\right)^{4} + 20$$
$$= 80\left(\frac{81}{256}\right) + 20$$
$$= 45.3125$$

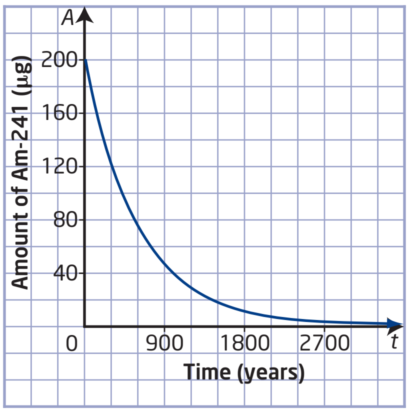
From the graph, the value of T when $m = 20$ is approximately 45. This matches the calculated value. Therefore, the transformed function that models the water temperature as it cools is $T(m) = 80\left(\frac{3}{4}\right)^{\frac{m}{5}} + 20$.

b) Based on the function $y = a(c)^{b(x-h)} + k$, the parameters of the transformed function are

- $b = \frac{1}{5}$, representing the interval of time, 5 min, over which a 25% decrease in temperature of the water occurs
- $a = 80$, representing the difference between the initial temperature of the heated cup of water and the air temperature of the room
- $h = 0$, representing the start time of the cooling process
- $k = 20$, representing the air temperature of the room

Your Turn

The radioactive element americium (Am) is used in household smoke detectors. Am-241 has a half-life of approximately 432 years. The average smoke detector contains 200 μg of Am-241.



a) What is the transformed exponential function that models the graph showing the radioactive decay of 200 μg of Am-241?

b) Identify how each of the parameters of the function relates to the transformed graph.

Did You Know?

In SI units, the symbol "μg" represents a microgram, or one millionth of a gram. In the medical field, the symbol "mcg" is used to avoid any confusion with milligrams (mg) in written prescriptions.

Key Ideas

- To sketch the graph of an exponential function of the form $y = a(c)^{b(x-h)} + k$, apply transformations to the graph of $y = c^x$, where $c > 0$. The transformations represented by a and b may be applied in any order before the transformations represented by h and k.
- The parameters a, b, h, and k in exponential functions of the form $y = a(c)^{b(x-h)} + k$ correspond to the following transformations:
  - a corresponds to a vertical stretch about the x-axis by a factor of $|a|$ and, if $a < 0$, a reflection in the x-axis.
  - b corresponds to a horizontal stretch about the y-axis by a factor of $\frac{1}{|b|}$ and, if $b < 0$, a reflection in the y-axis.
  - h corresponds to a horizontal translation left or right.
  - k corresponds to a vertical translation up or down.
- Transformed exponential functions can be used to model real-world applications of exponential growth or decay.

Check Your Understanding

Practise

1. Match each function with the corresponding transformation of $y = 3^x$.

a) $y = 2(3)^x$ **b)** $y = 3^{x-2}$

c) $y = 3^x + 4$ **d)** $y = 3^{\frac{x}{5}}$

A translation up

B horizontal stretch

C vertical stretch

D translation right

2. Match each function with the corresponding transformation of $y = \left(\frac{3}{5}\right)^x$.

a) $y = \left(\frac{3}{5}\right)^{x+1}$ **b)** $y = -\left(\frac{3}{5}\right)^x$

c) $y = \left(\frac{3}{5}\right)^{-x}$ **d)** $y = \left(\frac{3}{5}\right)^x - 2$

A reflection in the x-axis

B reflection in the y-axis

C translation down

D translation left

3. For each function, identify the parameters a, b, h, and k and the type of transformation that corresponds to each parameter.

a) $f(x) = 2(3)^x - 4$

b) $g(x) = 6^{x-2} + 3$

c) $m(x) = -4(3)^{x+5}$

d) $y = \left(\frac{1}{2}\right)^{3(x-1)}$

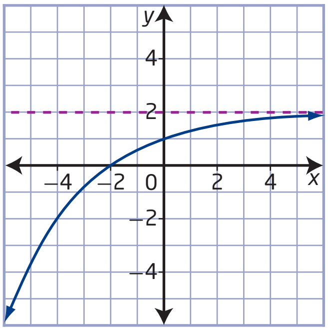
e) $n(x) = -\frac{1}{2}(5)^{2(x-4)} + 3$

f) $y = -\left(\frac{2}{3}\right)^{2x-2}$

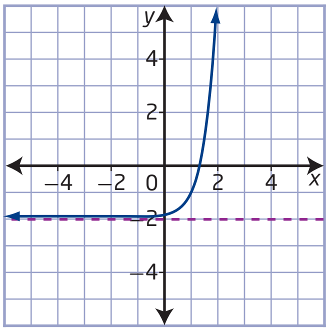
g) $y = 1.5(0.75)^{\frac{x-4}{2}} - \frac{5}{2}$

4. Without using technology, match each graph with the corresponding function. Justify your choice.

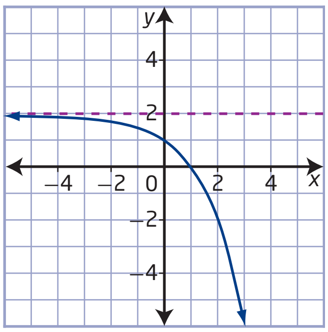
a)



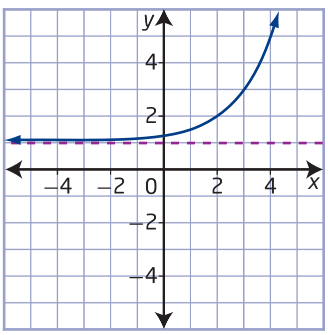
b)



c)



d)



A $y = 3^{2(x-1)} - 2$

B $y = 2^{x-2} + 1$

C $y = -\left(\frac{1}{2}\right)^{\frac{1}{2}x} + 2$

D $y = -\frac{1}{2}(4)^{\frac{1}{2}(x+1)} + 2$

5. The graph of $y = 4^x$ is transformed to obtain the graph of $y = \frac{1}{2}(4)^{-(x-3)} + 2$.

a) What are the parameters and corresponding transformations?

b) Copy and complete the table.

$y = 4^x$	$y = 4^{-x}$	$y = \frac{1}{2}(4)^{-x}$	$y = \frac{1}{2}(4)^{-(x-3)} + 2$
$\left(-2, \frac{1}{16}\right)$			
$\left(-1, \frac{1}{4}\right)$			
(0, 1)			
(1, 4)			
(2, 16)			

c) Sketch the graph of $y = \frac{1}{2}(4)^{-(x-3)} + 2$.

d) Identify the domain, range, equation of the horizontal asymptote, and any intercepts for the function $y = \frac{1}{2}(4)^{-(x-3)} + 2$.

6. For each function,

i) state the parameters a, b, h, and k

ii) describe the transformation that corresponds to each parameter

iii) sketch the graph of the function

iv) identify the domain, range, equation of the horizontal asymptote, and any intercepts

a) $y = 2(3)^x + 4$

b) $m(r) = -(2)^{r-3} + 2$

c) $y = \frac{1}{3}(4)^{x+1} + 1$

d) $n(s) = -\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{4}s} - 3$

Apply

7. Describe the transformations that must be applied to the graph of each exponential function $f(x)$ to obtain the transformed function. Write each transformed function in the form $y = a(c)^{b(x-h)} + k$.

a) $f(x) = \left(\frac{1}{2}\right)^x$, $y = f(x - 2) + 1$

b) $f(x) = 5^x$, $y = -0.5f(x - 3)$

c) $f(x) = \left(\frac{1}{4}\right)^x$, $y = -f(3x) + 1$

d) $f(x) = 4^x$, $y = 2f\left(-\frac{1}{3}(x - 1)\right) - 5$

8. For each pair of exponential functions in #7, sketch the original and transformed functions on the same set of coordinate axes. Explain your procedure.

9. The persistence of drugs in the human body can be modelled using an exponential function. Suppose a new drug follows the model $M(h) = M_0(0.79)^{\frac{h}{3}}$, where M is the mass, in milligrams, of drug remaining in the body; M_0 is the mass, in milligrams, of the dose taken; and h is the time, in hours, since the dose was taken.
   a) Explain the roles of the numbers 0.79 and $\frac{1}{3}$.
   b) A standard dose is 100 mg. Sketch the graph showing the mass of the drug remaining in the body for the first 48 h.
   c) What does the M-intercept represent in this situation?
   d) What are the domain and range of this function?

10. The rate at which liquids cool can be modelled by an approximation of Newton's law of cooling, $T(t) = (T_i - T_f)(0.9)^{\frac{t}{5}} + T_f$, where T_f represents the final temperature, in degrees Celsius; T_i represents the initial temperature, in degrees Celsius; and t represents the elapsed time, in minutes. Suppose a cup of coffee is at an initial temperature of 95 °C and cools to a temperature of 20 °C.
   a) State the parameters a, b, h, and k for this situation. Describe the transformation that corresponds to each parameter.
   b) Sketch a graph showing the temperature of the coffee over a period of 200 min.
   c) What is the approximate temperature of the coffee after 100 min?
   d) What does the horizontal asymptote of the graph represent?

11. A biologist places agar, a gel made from seaweed, in a Petri dish and infects it with bacteria. She uses the measurement of the growth ring to estimate the number of bacteria present. The biologist finds that the bacteria increase in population at an exponential rate of 20% every 2 days.
   a) If the culture starts with a population of 5000 bacteria, what is the transformed exponential function in the form $P = a(c)^{bx}$ that represents the population, P, of the bacteria over time, x, in days?
   b) Describe the parameters used to create the transformed exponential function.
   c) Graph the transformed function and use it to predict the bacteria population after 9 days.

12. Living organisms contain carbon-12 (C-12), which does not decay, and carbon-14 (C-14), which does. When an organism dies, the amount of C-14 in its tissues decreases exponentially with a half-life of about 5730 years.
   a) What is the transformed exponential function that represents the percent, P, of C-14 remaining after t years?
   b) Graph the function and use it to determine the approximate age of a dead organism that has 20% of the original C-14 present in its tissues.

Did You Know?

Carbon dating can only be used to date organic material, or material from once-living things. It is only effective in dating organisms that lived up to about 60 000 years ago.

Extend

13. On Monday morning, Julia found that a colony of bacteria covered an area of 100 cm^2 on the agar. After 10 h, she found that the area had increased to 200 cm^2. Assume that the growth is exponential.

a) By Tuesday morning (24 h later), what area do the bacteria cover?

b) Consider Earth to be a sphere with radius 6378 km. How long would these bacteria take to cover the surface of Earth?

14. Fifteen years ago, the fox population of a national park was 325 foxes. Today, it is 650 foxes. Assume that the population has experienced exponential growth.

a) Project the fox population in 20 years.

b) What is one factor that might slow the growth rate to less than exponential? Is exponential growth healthy for a population? Why or why not?

Create Connections

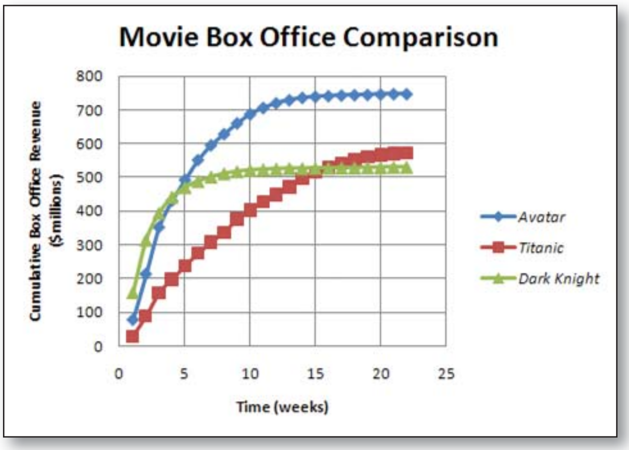
C1 The graph of an exponential function of the form $y = c^x$ does not have an x-intercept. Explain why this occurs using an example of your own.

C2 a) Which parameters of an exponential function affect the x-intercept of the graph of the function? Explain.

b) Which parameters of an exponential function affect the y-intercept of the graph of the function? Explain.

Project Corner — Modelling a Curve

It is not easy to determine the best mathematical model for real data. In many situations, one model works best for a limited period of time, and then another model is better. Work with a partner. Let x represent the time, in weeks, and let y represent the cumulative box office revenue, in millions of dollars.



- The curves for *Avatar* and *Dark Knight* appear to have a horizontal asymptote. What do you think this represents in this context? Do you think the curve for *Titanic* will eventually exhibit this characteristic as well? Explain.
- Consider the curve for *Titanic*.
  - If the vertex is located at (22, 573), determine a quadratic function of the form $y = a(x - h)^2 + k$ that might model this portion of the curve.
  - Suppose that the curve has a horizontal asymptote with equation $y = 600$. Determine an exponential function of the form $y = -35(0.65)^{0.3(x-h)} + k$ that might model the curve.
  - Which type of function do you think better models this curve? Explain.

7.3

Solving Exponential Equations

Focus on...

- determining the solution of an exponential equation in which the bases are powers of one another
- solving problems that involve exponential growth or decay
- solving problems that involve the application of exponential equations to loans, mortgages, and investments

Banks, credit unions, and investment firms often have financial calculators on their Web sites. These calculators use a variety of formulas, some based on exponential functions, to help users calculate amounts such as annuity values or compound interest in savings accounts. For compound interest calculators, users must input the dollar amount of the initial deposit; the amount of time the money is deposited, called the term; and the interest rate the financial institution offers.

Did You Know?

In 1935, the first Franco-Albertan credit union was established in Calgary.

Investigate the Different Ways to Express Exponential Expressions

Materials

- graphing technology

1. a) Copy the table into your notebook and complete it by

- substituting the value of n into each exponential expression
- using your knowledge of exponent laws to rewrite each expression as an equivalent expression with base 2

n	$\left(\frac{1}{2}\right)^n$	2^n	4^n
-2	$\left(\frac{1}{2}\right)^{-2} = (2^{-1})^{-2}$ $= 2^2$		
-1			
0			
1			
2			$4^2 = (2^2)^2$ $= 2^4$

b) What patterns do you observe in the equivalent expressions? Discuss your findings with a partner.

2. For each exponential expression in the column for 2^n, identify the equivalent exponential expressions with different bases in the other expression columns.

Reflect and Respond

3. a) Explain how to rewrite the **exponential equation** $2^x = 8^{x-1}$ so that the bases are the same.
   b) Describe how you could use this information to solve for x. Then, solve for x.
   c) Graph the exponential functions on both sides of the equation in part a) on the same set of axes. Explain how the point of intersection of the two graphs relates to the solution you determined in part b).
4. a) Consider the exponential equation $3^x = 4^{2x-1}$. Can this equation be solved in the same way as the one in step 3a)? Explain.
   b) What are the limitations when solving exponential equations that have terms with different bases?

exponential equation

- an equation that has a variable in an exponent

Link the Ideas

Exponential expressions can be written in different ways. It is often useful to rewrite an exponential expression using a different base than the one that is given. This is helpful when solving exponential equations because the exponents of exponential expressions with the same base can be equated to each other. For example,

$$4^{2x} = 8^{x+1}$$
$$(2^2)^{2x} = (2^3)^{x+1} \quad \text{Express the base on each side as a power of 2.}$$
$$2^{4x} = 2^{3x+3}$$

Since the bases on both sides of the equation are now the same, the exponents must be equal.

Is this statement true for all bases? Explain.

$$4x = 3x + 3 \quad \text{Equate exponents.}$$
$$x = 3$$

This method of solving an exponential equation is based on the property that if $c^x = c^y$, then $x = y$, for $c \neq -1, 0, 1$.

Example 1

Change the Base of Powers

Rewrite each expression as a power with a base of 3.

a) 27 **b)** 9^2 **c)** $27^{\frac{1}{3}}\left(\sqrt[3]{81}\right)^2$

Solution

a) $27 = 3^3$ 27 is the third power of 3.

b) $9^2 = (3^2)^2$ Write 9 as 3^2.

$= 3^4$ Apply the power of a power law.

c)
$$
\begin{aligned}
27^{\frac{1}{3}}\left(\sqrt[3]{81}\right)^2 &= 27^{\frac{1}{3}}\left(81^{\frac{2}{3}}\right) && \text{Write the radical in exponential form.}\\
&= (3^3)^{\frac{1}{3}}(3^4)^{\frac{2}{3}} && \text{Express the bases as powers of 3.}\\
&= 3^1\left(3^{\frac{8}{3}}\right) && \text{Apply the power of a power law.}\\
&= 3^{1+\frac{8}{3}} && \text{Apply the product of powers law.}\\
&= 3^{\frac{11}{3}} && \text{Simplify.}
\end{aligned}
$$

Your Turn

Write each expression as a power with base 2.

a) 4^3 **b)** $\frac{1}{8}$ **c)** $8^{\frac{2}{3}}\left(\sqrt{16}\right)^3$

Example 2

Solve an Equation by Changing the Base

Solve each equation.

a) $4^{x+2} = 64^x$ **b)** $4^{2x} = 8^{2x-3}$

Solution

a) **Method 1: Apply a Change of Base**

$$
\begin{aligned}
4^{x+2} &= 64^x\\
4^{x+2} &= (4^3)^x && \text{Express the base on the right side as a power with base 4.}\\
4^{x+2} &= 4^{3x} && \text{Apply the power of a power law.}
\end{aligned}
$$

Since both sides are single powers of the same base, the exponents must be equal.

Equate the exponents.

$$
\begin{aligned}
x + 2 &= 3x\\
2 &= 2x && \text{Isolate the term containing } x.\\
x &= 1
\end{aligned}
$$

Check:

Left Side: $4^{x+2} = 4^{1+2} = 4^3 = 64$

Right Side: $64^x = 64^1 = 64$

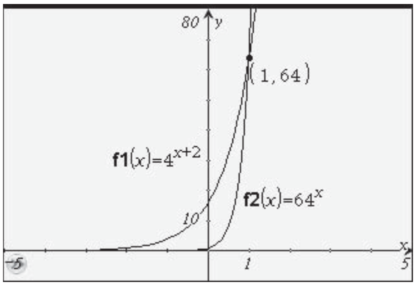
Left Side = Right Side

The solution is $x = 1$.

Method 2: Use a Graphing Calculator

Enter the left side of the equation as one function and the right side as another function. Identify where the graphs intersect using the intersection feature.

You may have to adjust the window settings to view the point of intersection.



The graphs intersect at the point (1, 64).
The solution is $x = 1$.

b)

$$
\begin{aligned}
4^{2x} &= 8^{2x-3} \\
(2^2)^{2x} &= (2^3)^{2x-3} && \text{Express the bases on both sides as powers of 2.} \\
2^{4x} &= 2^{6x-9} && \text{Apply the power of a power law.} \\
4x &= 6x - 9 && \text{Equate the exponents.} \\
-2x &= -9 && \text{Isolate the term containing } x. \\
x &= \frac{9}{2} && \text{Solve for } x.
\end{aligned}
$$

Check:

Left Side: $4^{2x} = 4^{2\left(\frac{9}{2}\right)} = 4^9 = 262\,144$

Right Side: $8^{2x-3} = 8^{2\left(\frac{9}{2}\right)-3} = 8^{9-3} = 8^6 = 262\,144$

Left Side = Right Side

The solution is $x = \frac{9}{2}$.

Your Turn

Solve. Check your answers using graphing technology.

a) $2^{4x} = 4^{x+3}$

b) $9^{4x} = 27^{x-1}$

Example 3

Solve Problems Involving Exponential Equations With Different Bases

Christina plans to buy a car. She has saved \$5000. The car she wants costs \$5900. How long will Christina have to invest her money in a term deposit that pays 6.12% per year, compounded quarterly, before she has enough to buy the car?

Solution

The formula for compound interest is $A = P(1 + i)^n$, where A is the amount of money at the end of the investment; P is the principal amount deposited; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

In this problem:

$A = 5900$

$P = 5000$

$i = 0.0612 \div 4$ or 0.0153

Divide the interest rate by 4 because interest is paid quarterly or four times a year.

Substitute the known values into the formula.

$$A = P(1 + r)^n$$
$$5900 = 5000(1 + 0.0153)^n$$
$$1.18 = 1.0153^n$$

You will learn how to solve equations like this algebraically when you study logarithms in Chapter 8.

The exponential equation consists of bases that cannot be changed into the same form without using more advanced mathematics.

Method 1: Use Systematic Trial

Use systematic trial to find the approximate value of n that satisfies this equation.

Substitute an initial guess into the equation and evaluate the result. Adjust the estimated solution according to whether the result is too high or too low.

Try $n = 10$.

Why choose whole numbers for n?

$1.0153^{10} = 1.1639\ldots$, which is less than 1.18.

The result is less than the left side of the equation, so try a value of $n = 14$.
$1.0153^{14} = 1.2368\ldots$, which is greater than 1.18.

The result is more than the left side of the equation, so try a value of $n = 11$.
$1.0153^{11} = 1.1817\ldots$, which is approximately equal to 1.18.

The number of compounding periods is approximately 11.

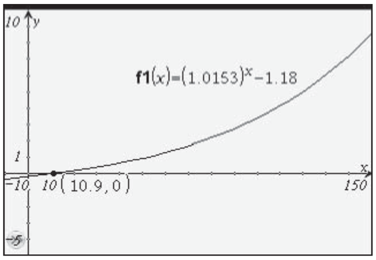
Since interest is paid quarterly, there are four compounding periods in each year. Therefore, it will take approximately $\frac{11}{4}$ or 2.75 years for Christina's investment to reach a value of \$5900.

Method 2: Use a Graphing Calculator

Enter the single function $y = 1.0153^x - 1.18$ and identify where the graph intersects the x-axis.

How is this similar to graphing the left side and right side of the equation and determining where the two graphs intersect?

You may have to adjust the window settings to view the point of intersection.



Use the features of the graphing calculator to show that the zero of the function is approximately 11.

Since interest is paid quarterly, there are four compounding periods in each year. Therefore, it will take approximately $\frac{11}{4}$ or 2.75 years for Christina's investment to reach a value of \$5900.

Your Turn

Determine how long \$1000 needs to be invested in an account that earns 8.3% compounded semi-annually before it increases in value to \$1490.

Key Ideas

- Some exponential equations can be solved directly if the terms on either side of the equal sign have the same base or can be rewritten so that they have the same base.
  - If the bases are the same, then equate the exponents and solve for the variable.
  - If the bases are different but can be rewritten with the same base, use the exponent laws, and then equate the exponents and solve for the variable.
- Exponential equations that have terms with bases that you cannot rewrite using a common base can be solved approximately. You can use either of the following methods:
  - Use systematic trial. First substitute a reasonable estimate for the solution into the equation, evaluate the result, and adjust the next estimate according to whether the result is too high or too low. Repeat this process until the sides of the equation are approximately equal.
  - Graph the functions that correspond to the expressions on each side of the equal sign, and then identify the value of x at the point of intersection, or graph as a single function and find the x-intercept.

Check Your Understanding

Practise

1. Write each expression with base 2.

   a) 4^6 b) 8^3

   c) $\left(\frac{1}{8}\right)^2$ d) 16

2. Rewrite the expressions in each pair so that they have the same base.

   a) 2^3 and 4^2

   b) 9^x and 27

   c) $\left(\frac{1}{2}\right)^{2x}$ and $\left(\frac{1}{4}\right)^{x-1}$

   d) $\left(\frac{1}{8}\right)^{x-2}$ and 16^x

3. Write each expression as a single power of 4.

   a) $(\sqrt{16})^2$ b) $\sqrt[3]{16}$

   c) $\sqrt{16}\,(\sqrt[3]{64})^2$ d) $(\sqrt{2})^8(\sqrt[4]{4})^4$

4. Solve. Check your answers using substitution.

   a) $2^{4x} = 4^{x+3}$ b) $25^{x-1} = 5^{3x}$

   c) $3^{w+1} = 9^{w-1}$ d) $36^{3m-1} = 6^{2m+5}$

5. Solve. Check your answers using graphing technology.

   a) $4^{3x} = 8^{x-3}$ b) $27^x = 9^{x-2}$

   c) $125^{2y-1} = 25^{y+4}$ d) $16^{2k-3} = 32^{k+3}$

6. Solve for x using systematic trial. Check your answers using graphing technology. Round answers to one decimal place.

   a) $2 = 1.07^x$ b) $3 = 1.1^x$

   c) $0.5 = 1.2^{x-1}$ d) $5 = 1.08^{x+2}$

7. Solve for t graphically. Round answers to two decimal places, if necessary.

   a) $100 = 10(1.04)^t$ b) $10 = \left(\frac{1}{2}\right)^{2t}$

   c) $12 = \left(\frac{1}{4}\right)^{\frac{t}{3}}$ d) $100 = 25\left(\frac{1}{2}\right)^{\frac{t}{4}}$

   e) $2^t = 3^{t-1}$ f) $5^{t-2} = 4^t$

   g) $8^{t+1} = 3^{t-1}$ h) $7^{2t+1} = 4^{t-2}$

Apply

8. If seafood is not kept frozen (below 0 °C), it will spoil due to bacterial growth. The relative rate of spoilage increases with temperature according to the model $R = 100(2.7)^{\frac{T}{8}}$, where T is the temperature, in degrees Celsius, and R is the relative spoilage rate.

   a) Sketch a graph of the relative spoilage rate R versus the temperature T from 0 °C to 25 °C.

   b) Use your graph to predict the temperature at which the relative spoilage rate doubles to 200.

   c) What is the relative spoilage rate at 15 °C?

   d) If the maximum acceptable relative spoilage rate is 500, what is the maximum storage temperature?

Did You Know?

The relative rate of spoilage for seafood is defined as the shelf life at 0° C divided by the shelf life at temperature T, in degrees Celsius.

9. A bacterial culture starts with 2000 bacteria and doubles every 0.75 h. After how many hours will the bacteria count be 32 000?

10. Simionie needs $7000 to buy a snowmobile, but only has $6000. His bank offers a GIC that pays an annual interest rate of 3.93%, compounded annually. How long would Simionie have to invest his money in the GIC to have enough money to buy the snowmobile?

Did You Know?

A Guaranteed Investment Certificate (GIC) is a secure investment that guarantees 100% of the original amount that is invested. The investment earns interest, at either a fixed or a variable rate, based on a predetermined formula.

11. A \$1000 investment earns interest at a rate of 8% per year, compounded quarterly.

a) Write an equation for the value of the investment as a function of time, in years.

b) Determine the value of the investment after 4 years.

c) How long will it take for the investment to double in value?

12. Cobalt-60 (Co-60) has a half-life of approximately 5.3 years.

a) Write an exponential function to model this situation.

b) What fraction of a sample of Co-60 will remain after 26.5 years?

c) How long will it take for a sample of Co-60 to decay to $\frac{1}{512}$ of its original mass?

13. A savings bond offers interest at a rate of 6.6% per year, compounded semi-annually. Suppose that you buy a \$500 bond.

a) Write an equation for the value of the investment as a function of time, in years.

b) Determine the value of the investment after 5 years.

c) How long will it take for the bond to triple in value?

14. Glenn and Arlene plan to invest money for their newborn grandson so that he has \$20 000 available for his education on his 18th birthday. Assuming a growth rate of 7% per year, compounded semi-annually, how much will Glenn and Arlene need to invest today?

Did You Know?

When the principal, P, needed to generate a future amount is unknown, you can rearrange the compound interest formula to isolate P: $P = A(1 + i)^{-n}$. In this form, the principal is referred to as the present value and the amount is referred to as the future value. Then, you can calculate the present value, PV, the amount that must be invested or borrowed today to result in a specific future value, FV, using the formula $PV = FV(1 + i)^{-n}$, where i is the interest rate per compounding period, expressed as a decimal value, and n is the number of compounding periods.

Extend

15. a) Solve each inequality.

i) $2^{3x} > 4^{x+1}$ **ii)** $81^x < 27^{2x+1}$

b) Use a sketch to help you explain how you can use graphing technology to check your answers.

c) Create an inequality involving an exponential expression. Solve the inequality graphically.

16. Does the equation $4^{2x} + 2(4^x) - 3 = 0$ have any real solutions? Explain your answer.

17. If $4^x - 4^{x-1} = 24$, what is the value of $(2^x)^x$?

18. The formula for calculating the monthly mortgage payment, PMT, for a property is $PMT = PV\left[\frac{i}{1 - (1 + i)^{-n}}\right]$, where PV is the present value of the mortgage; i is the interest rate per compounding period, as a decimal; and n is the number of payment periods. To buy a house, Tyseer takes out a mortgage worth \$150 000 at an equivalent monthly interest rate of 0.25%. He can afford monthly mortgage payments of \$831.90. Assuming the interest rate and monthly payments stay the same, how long will it take Tyseer to pay off the mortgage?

Create Connections

C1 a) Explain how you can write 16^2 with base 4.

b) Explain how you can write 16^2 with two other, different, bases.

C2 The steps for solving the equation $16^{2x} = 8^{x-3}$ are shown below, but in a jumbled order.

$$2^{8x} = 2^{3x-9}$$
$$16^{2x} = 8^{x-3}$$
$$x = -\frac{9}{5}$$
$$8x = 3x - 9$$
$$(2^4)^{2x} = (2^3)^{x-3}$$
$$5x = -9$$

a) Copy the steps into your notebook, rearranged in the correct order.

b) Write a brief explanation beside each step.

Chapter 7 Review

7.1 Characteristics of Exponential Functions, pages 334–345

1. Match each item in set A with its graph from set B.

Set A

a) The population of a country, in millions, grows at a rate of 1.5% per year.

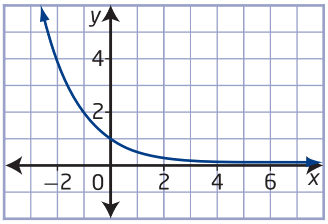
b) $y = 10^x$

c) Tungsten-187 is a radioactive isotope that has a half-life of 1 day.

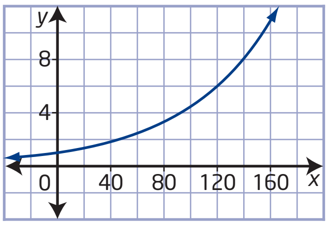
d) $y = 0.2^x$

Set B

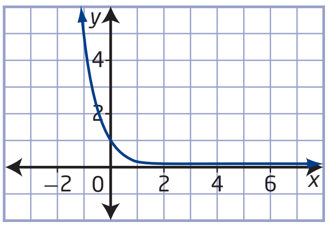
A



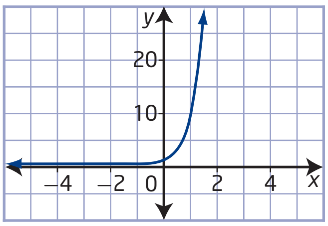
B



C



D

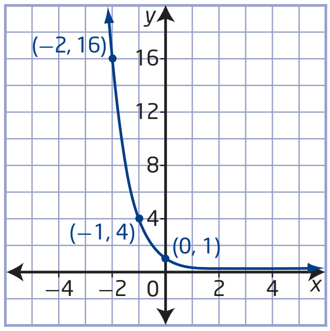


2. Consider the exponential function $y = 0.3^x$.

a) Make a table of values and sketch the graph of the function.

b) Identify the domain, range, intercepts, and intervals of increase or decrease, as well as any asymptotes.

3. What exponential function in the form $y = c^x$ is represented by the graph shown?



4. The value, v, of a dollar invested for t years at an annual interest rate of 3.25% is given by $v = 1.0325^t$.

a) Explain why the base of the exponential function is 1.0325.

b) What will be the value of $1 if it is invested for 10 years?

c) How long will it take for the value of the dollar invested to reach $2?

7.2 Transformations of Exponential Functions, page 346–357

5. The graph of $y = 4^x$ is transformed to obtain the graph of $y = -2(4)^{3(x-1)} + 2$.

a) What are the parameters and corresponding transformations?

b) Copy and complete the table.

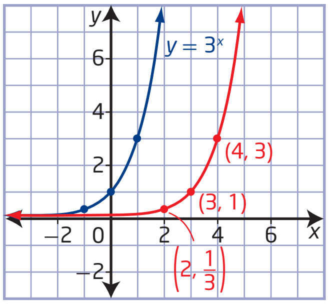
Transformation	Parameter Value	Function Equation
horizontal stretch		
vertical stretch		
translation left/right		
translation up/down		

c) Sketch the graph of $y = -2(4)^{3(x-1)} + 2$.

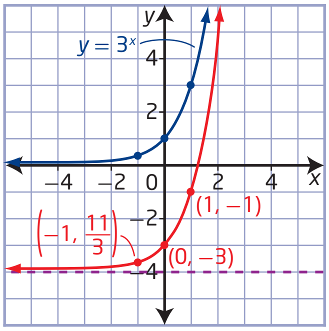
d) Identify the domain, range, equation of the horizontal asymptote, and any intercepts for the function $y = -2(4)^{3(x-1)} + 2$.

6. Identify the transformation(s) used in each case to transform the base function $y = 3^x$.

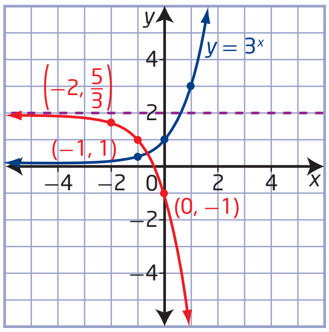
a)



b)



c)



7. Write the equation of the function that results from each set of transformations, and then sketch the graph of the function.

a) $f(x) = 5^x$ is stretched vertically by a factor of 4, stretched horizontally by a factor of $\frac{1}{2}$, reflected in the y-axis, and translated 1 unit up and 4 units to the left.

b) $g(x) = \left(\frac{1}{2}\right)^x$ is stretched horizontally by a factor of $\frac{1}{4}$, stretched vertically by a factor of 3, reflected in the x-axis, and translated 2 units to the right and 1 unit down.

8. The function $T = 190\left(\frac{1}{2}\right)^{\frac{1}{10}t}$ can be used to determine the length of time, t, in hours, that milk of a certain fat content will remain fresh. T is the storage temperature, in degrees Celsius.

a) Describe how each of the parameters in the function transforms the base function $T = \left(\frac{1}{2}\right)^t$.

b) Graph the transformed function.

c) What are the domain and range for this situation?

d) How long will milk keep fresh at 22 °C?

7.3 Solving Exponential Equations, pages 358–365

9. Write each as a power of 6.

a) 36

b) $\frac{1}{36}$

c) $\left(\sqrt[3]{216}\right)^5$

10. Solve each equation. Check your answers using graphing technology.

a) $3^{5x} = 27^{x-1}$

b) $\left(\frac{1}{8}\right)^{2x+1} = 32^{x-3}$

11. Solve for x. Round answers to two decimal places.

a) $3^{x-2} = 5^x$

b) $2^{x-2} = 3^{x+1}$

12. Nickel-65 (Ni-65) has a half-life of 2.5 h.

a) Write an exponential function to model this situation.

b) What fraction of a sample of Ni-65 will remain after 10 h?

c) How long will it take for a sample of Ni-65 to decay to $\frac{1}{1024}$ of its original mass?

Chapter 7 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. Consider the exponential functions $y = 2^x$, $y = \left(\frac{2}{3}\right)^x$, and $y = 7^x$. Which value of x results in the same y-value for each?

A -1

B 0

C 1

D There is no such value of x.

2. Which statement describes how to transform the function $y = 3^x$ into $y = 3^{\frac{1}{4}(x-5)} - 2$?

A stretch vertically by a factor of $\frac{1}{4}$ and translate 5 units to the left and 2 units up

B stretch horizontally by a factor of $\frac{1}{4}$ and translate 2 units to the right and 5 units down

C stretch horizontally by a factor of 4 and translate 5 units to the right and 2 units down

D stretch horizontally by a factor of 4 and translate 2 units to the left and 5 units up

3. An antique automobile was found to double in value every 10 years. If the current value is \$100 000, what was the value of the vehicle 20 years ago?

A \$50 000 **B** \$25 000

C \$12 500 **D** \$5000

4. What is $\frac{2^9}{(4^3)^2}$ expressed as a power of 2?

A 2^{-3}

B 2^3

C 2^1

D 2^{-1}

5. The intensity, I, in lumens, of light passing through the glass of a pair of sunglasses is given by the function $I(x) = I_0(0.8)^x$, where x is the thickness of the glass, in millimetres, and I_0 is the intensity of light entering the glasses. Approximately how thick should the glass be so that it will block 25% of the light entering the sunglasses?

A 0.7 mm

B 0.8 mm

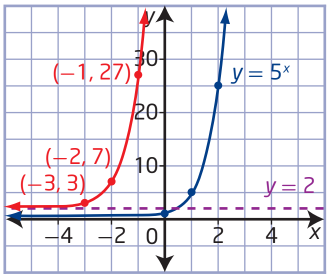
C 1.1 mm

D 1.3 mm

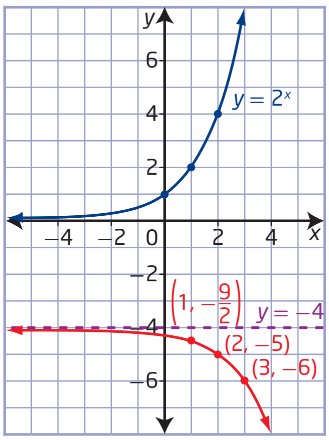
Short Answer

6. Determine the function that represents each transformed graph.

a)



b)



7. Sketch and label the graph of each exponential function.

   a) $y = \frac{1}{2}(3)^x + 2$

   b) $y = -2\left(\frac{3}{2}\right)^{x-1} - 2$

   c) $y = 3^{2(x+3)} - 4$

8. Consider the function $g(x) = 2(3)^{x+3} - 4$.

   a) Determine the base function for $g(x)$ and describe the transformations needed to transform the base function to $g(x)$.

   b) Graph the function $g(x)$.

   c) Identify the domain, the range, and the equation of the horizontal asymptote for $g(x)$.

9. Solve for x.

   a) $3^{2x} = 9^{\frac{1}{2}(x-4)}$

   b) $27^{x-4} = 9^{x+3}$

   c) $1024^{2x-1} = 16^{x+4}$

10. Solve each equation using graphing technology. Round answers to one decimal place.

    a) $3 = 1.12^x$

    b) $2.7 = 0.3^{2x-1}$

Extended Response

11. According to a Statistics Canada report released in 2010, Saskatoon had the fastest-growing population in Canada, with an annual growth rate of 2.77%.

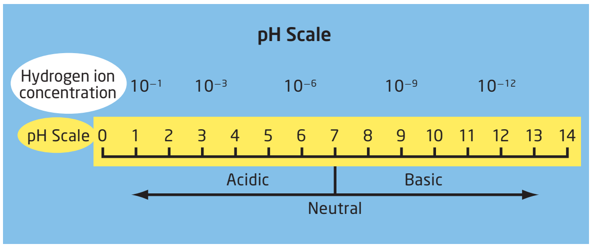
    a) If the growth rate remained constant, by what factor would the population have been multiplied after 1 year?

    b) What function could be used to model this situation?

    c) What are the domain and range of the function for this situation?

    d) At this rate, approximately how long would it take for Saskatoon's population to grow by 25%?

12. The measure of the acidity of a solution is called its pH. The pH of swimming pools needs to be checked regularly. This is done by measuring the concentration of hydrogen ions (H^+) in the water. The relationship between the hydrogen ion concentration, H, in moles per litre (mol/L), is $H(P) = \left(\frac{1}{10}\right)^P$, where P is the pH.



    a) Sketch the graph of this function.

    b) Water with a pH of less than 7.0 is acidic. What is the hydrogen ion concentration for a pH of 7.0?

    c) Water in a swimming pool should have a pH of between 7.0 and 7.6. What is the equivalent range of hydrogen ion concentration?

13. Lucas is hoping to take a vacation after he finishes university. To do this, he estimates he needs \$5000. Lucas is able to finish his last year of university with \$3500 in an investment that pays 8.4% per year, compounded quarterly. How long will Lucas have to wait before he has enough money to take the vacation he wants?

14. A computer, originally purchased for \$3000, depreciates in value according to the function $V(t) = 3000\left(\frac{1}{2}\right)^{\frac{t}{3}}$, where V is the value, in dollars, of the computer at any time, t, in years. Approximately how long will it take for the computer to be worth 10% of its purchase price?

CHAPTER 8

Logarithmic Functions

Logarithms were developed over 400 years ago, and they still have numerous applications in the modern world. Logarithms allow you to solve any exponential equation. Logarithmic scales use manageable numbers to represent quantities in science that vary over vast ranges, such as the energy of an earthquake or the pH of a solution. Logarithmic spirals model the spiral arms of a galaxy, the curve of animal horns, the shape of a snail, the growth of certain plants, the arms of a hurricane, and the approach of a hawk to its prey.

In this chapter, you will learn what logarithms are, how to represent them, and how to use them to model situations and solve problems.

Did You Know?

Logarithms were developed independently by John Napier (1550–1617), from Scotland, and Jobst Bürgi (1552–1632), from Switzerland. Since Napier published his work first, he is given the credit. Napier was also the first to use the decimal point in its modern context.

Logarithms were developed before exponents were used. It was not until the end of the seventeenth century that mathematicians recognized that logarithms are exponents.

Key Terms

logarithmic function	common logarithm
logarithm	logarithmic equation

Career Link

A radiologist is a physician trained in diagnosing and treating disease and injury using medical-imaging techniques. Radiologists use X-rays, computerized tomography (CT), magnetic resonance imaging (MRI), positron emission tomography (PET), fusion imaging, and ultrasound. Since some of these imaging techniques involve the use of radioactive isotopes, radiologists have special training in radiation physics, the effects of radiation on the body, and radiation safety and protection.

Web Link

To learn more about a career in radiology, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

8.1

Understanding Logarithms

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

Do you have a favourite social networking site? Some social networking sites can be modelled by an exponential function, where the number of users is a function of time. You can use the exponential function to predict the number of users accessing the site at a certain time.

What if you wanted to predict the length of time required for a social networking site to be accessed by a certain number of users? In this type of relationship, the length of time is a function of the number of users. This situation can be modelled by using the inverse of an exponential function.

Investigate Logarithms

Materials

- grid paper

1. Use a calculator to determine the decimal approximation of the exponent, x, in each equation, to one decimal place.

a) $10^x = 0.5$ **b)** $10^x = 4$ **c)** $10^x = 8$

2. a) Copy and complete the table of values for the exponential function $y = 10^x$ and then draw the graph of the function.

x	–1		0			1
y		0.5		4	8	

b) Identify the following characteristics of the graph.

i) the domain

ii) the range

iii) the x-intercept, if it exists

iv) the y-intercept, if it exists

v) the equation of any asymptotes

3. a) Copy and complete the table of values for $x = 10^y$, which is the inverse of $y = 10^x$. Then, draw the graph of the inverse function.

x		0.5		4	8	
y	−1		0			1

b) Identify the following characteristics of the inverse graph.

i) the domain

ii) the range

iii) the x-intercept, if it exists

iv) the y-intercept, if it exists

v) the equation of the asymptote

4. Use the log function on a calculator to find the decimal approximation of each expression, to three decimal places. What do you notice about these values?

a) log 0.5

b) log 4

c) log 8

Reflect and Respond

5. Explain how the graph of the exponential function $y = 10^x$ and its inverse graph are related.

6. Does the inverse graph represent a function? Explain.

7. Points on the inverse graph are of the form $(x, \log x)$. Explain the meaning of $\log x$.

Link the Ideas

For the exponential function $y = c^x$, the inverse is $x = c^y$. This inverse is also a function and is called a **logarithmic function**. It is written as $y = \log_c x$, where c is a positive number other than 1.

Logarithmic Form **Exponential Form**

exponent

$\log_c x = y$ $c^y = x$

base

Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a common logarithm, you do not need to write the base. For example, log 3 means $\log_{10} 3$.

logarithmic function

- a function of the form $y = \log_c x$, where $c > 0$ and $c \neq 1$, that is the inverse of the exponential function $y = c^x$

logarithm

- an exponent
- in $x = c^y$, y is called the logarithm to base c of x

common logarithm

- a logarithm with base 10

Example 1

Evaluating a Logarithm

Evaluate.

a) $\log_7 49$ **b)** $\log_6 1$ **c)** $\log 0.001$ **d)** $\log_2 \sqrt{8}$

Solution

a) The logarithm is the exponent that must be applied to base 7 to obtain 49. Determine the value by inspection.
Since $7^2 = 49$, the value of the logarithm is 2.
Therefore, $\log_7 49 = 2$.

b) The logarithm is the exponent that must be applied to base 6 to obtain 1.
Since, $6^0 = 1$, the value of the logarithm is 0.
Therefore, $\log_6 1 = 0$.

What is the value of any logarithm with an argument of 1? Why?

c) This is a common logarithm. You need to find the exponent that must be applied to base 10 to obtain 0.001.
Let $\log_{10} 0.001 = x$. Express in exponential form.

$$10^x = 0.001$$
$$10^x = \frac{1}{1000}$$
$$10^x = \frac{1}{10^3}$$
$$10^x = 10^{-3}$$
$$x = -3$$

Therefore, $\log 0.001 = -3$.

d) The logarithm is the exponent that must be applied to base 2 to obtain $\sqrt{8}$. Let $\log_2 \sqrt{8} = x$. Express in exponential form.

$$2^x = \sqrt{8}$$
$$2^x = \sqrt{2^3}$$
$$2^x = (2^3)^{\frac{1}{2}}$$

Express the radical as a power with a rational exponent.

$$2^x = 2^{\frac{3}{2}}$$
$$x = \frac{3}{2}$$

Therefore, $\log_2 \sqrt{8} = \frac{3}{2}$.

Your Turn

Evaluate.

a) $\log_2 32$
b) $\log_9 \sqrt[5]{81}$
c) $\log 1\ 000\ 000$
d) $\log_3 9\sqrt{3}$

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

If $c > 0$ and $c \neq 1$, then

Why does c have these restrictions?

- $\log_c 1 = 0$ since in exponential form $c^0 = 1$
- $\log_c c = 1$ since in exponential form $c^1 = c$
- $\log_c c^x = x$ since in exponential form $c^x = c^x$
- $c^{\log_c x} = x$, $x > 0$, since in logarithmic form $\log_c x = \log_c x$

The last two results are sometimes called the inverse properties, since logarithms and powers are inverse mathematical operations that undo each other. In $\log_c c^x = x$, the logarithm of a power with the same base equals the exponent, x. In $c^{\log_c x} = x$, a power raised to the logarithm of a number with the same base equals that number, x.

Example 2

Determine an Unknown in an Expression in Logarithmic Form

Determine the value of x.

a) $\log_5 x = -3$

b) $\log_x 36 = 2$

c) $\log_{64} x = \frac{2}{3}$

Solution

a)
$$\begin{aligned} \log_5 x &= -3 \\ 5^{-3} &= x \\ \frac{1}{125} &= x \end{aligned}$$

Express in exponential form.

b)
$$\begin{aligned} \log_x 36 &= 2 \\ x^2 &= 36 \\ x &= \pm\sqrt{36} \end{aligned}$$

Express in exponential form.

Since the base of a logarithm must be greater than zero, $x = -6$ is not an acceptable answer. So, $x = 6$.

c)
$$\begin{aligned} \log_{64} x &= \frac{2}{3} \\ 64^{\frac{2}{3}} &= x \\ (\sqrt[3]{64})^2 &= x \\ 4^2 &= x \\ 16 &= x \end{aligned}$$

Express in exponential form.

Your Turn

Determine the value of x.

a) $\log_4 x = -2$

b) $\log_{16} x = -\frac{1}{4}$

c) $\log_x 9 = \frac{2}{3}$

Example 3

Graph the Inverse of an Exponential Function

a) State the inverse of $f(x) = 3^x$.

b) Sketch the graph of the inverse. Identify the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

Solution

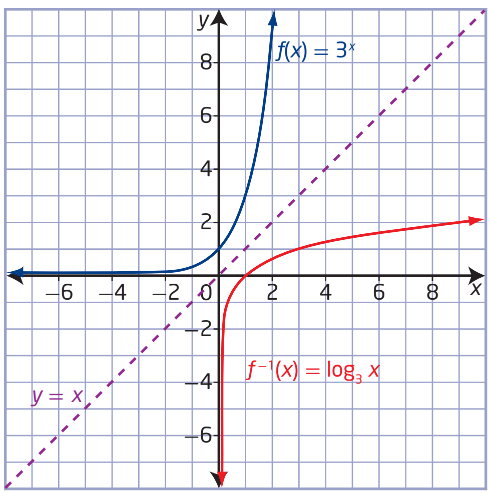
a) The inverse of $y = f(x) = 3^x$ is $x = 3^y$ or, expressed in logarithmic form, $y = \log_3 x$. Since the inverse is a function, it can be written in function notation as $f^{-1}(x) = \log_3 x$.

How do you know that $y = \log_3 x$ is a function?

b) Set up tables of values for both the exponential function, $f(x)$, and its inverse, $f^{-1}(x)$. Plot the points and join them with a smooth curve.

$f(x) = 3^x$	
x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

$f^{-1}(x) = \log_3 x$	
x	y
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



How are the values of x and y related in these two functions? Explain.

The graph of the inverse, $f^{-1}(x) = \log_3 x$, is a reflection of the graph of $f(x) = 3^x$ about the line $y = x$. For $f^{-1}(x) = \log_3 x$,

- the domain is $\{x \mid x > 0, x \in R\}$ and the range is $\{y \mid y \in R\}$
- the x-intercept is 1
- there is no y-intercept
- the vertical asymptote, the y-axis, has equation $x = 0$; there is no horizontal asymptote

How do the characteristics of $f^{-1}(x) = \log_3 x$ compare to the characteristics of $f(x) = 3^x$?

Your Turn

a) Write the inverse of $f(x) = \left(\frac{1}{2}\right)^x$.

b) Sketch the graphs of $f(x)$ and its inverse. Identify the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

Example 4

Estimate the Value of a Logarithm

Without using technology, estimate the value of $\log_2 14$, to one decimal place.

Solution

The logarithm is the exponent that must be applied to base 2 to obtain 14.

Since $2^3 = 8$, $\log_2 8 = 3$.
Also, $2^4 = 16$, so $\log_2 16 = 4$.
Since 14 is closer to 16 than to 8, try an estimate of 3.7.

Then, $2^{3.7} \approx 13$, so $\log_2 13 \approx 3.7$. This is less than $\log_2 14$.

Try 3.8. Then, $2^{3.8} \approx 14$, so $\log_2 14 \approx 3.8$.

Your Turn

Without using technology, estimate the value of $\log_3 50$, to one decimal place.

Example 5

An Application of Logarithms

In 1935, American seismologist Charles R. Richter developed a scale formula for measuring the magnitude of earthquakes. The Richter magnitude, M, of an earthquake is defined as $M = \log \frac{A}{A_0}$, where A is the amplitude of the ground motion, usually in microns, measured by a sensitive seismometer, and A_0 is the amplitude, corrected for the distance to the actual earthquake, that would be expected for a "standard" earthquake.


Naikoon Provincial Park, Haida Gwaii

a) In 1946, an earthquake struck Vancouver Island off the coast of British Columbia. It had an amplitude that was $10^{7.3}$ times A_0. What was the earthquake's magnitude on the Richter scale?

b) The strongest recorded earthquake in Canada struck Haida Gwaii, off the coast of British Columbia, in 1949. It had a Richter reading of 8.1. How many times as great as A_0 was its amplitude?

c) Compare the seismic shaking of the 1949 Haida Gwaii earthquake with that of the earthquake that struck Vancouver Island in 1946.

Did You Know?

A "standard" earthquake has amplitude of 1 micron, or 0.0001 cm, and magnitude 0. Each increase of 1 unit on the Richter scale is equivalent to a tenfold increase in the intensity of an earthquake.

Solution

a) Since the amplitude of the Vancouver Island earthquake was $10^{7.3}$ times A_0, substitute $10^{7.3}A_0$ for A in the formula $M = \log \frac{A}{A_0}$.

$$M = \log\left(\frac{10^{7.3}\overset{1}{\cancel{A_0}}}{\underset{1}{\cancel{A_0}}}\right)$$

$$M = \log 10^{7.3}$$

$$M = 7.3$$ $\log_c c^x = x$, since in exponential form $c^x = c^x$.

The Vancouver Island earthquake had magnitude of 7.3 on the Richter scale.

b) Substitute 8.1 for M in the formula $M = \log \frac{A}{A_0}$ and express A in terms of A_0.

$$8.1 = \log \frac{A}{A_0}$$

$$10^{8.1} = \frac{A}{A_0}$$ Write in exponential form.

$$10^{8.1}A_0 = A$$

$$125\ 892\ 541A_0 \approx A$$

The amplitude of the Haida Gwaii earthquake was approximately 126 million times the amplitude of a standard earthquake.

c) Compare the amplitudes of the two earthquakes.

$$\frac{\text{amplitude of Haida Gwaii earthquake}}{\text{amplitude of Vancouver Island earthquake}} = \frac{10^{8.1}\cancelto{1}{A_0}}{10^{7.3}\cancelto{1}{A_0}}$$
$$= \frac{10^{8.1}}{10^{7.3}}$$
$$\approx 6.3$$

The Haida Gwaii earthquake created shaking 6.3 times as great in amplitude as the Vancouver Island earthquake.

Your Turn

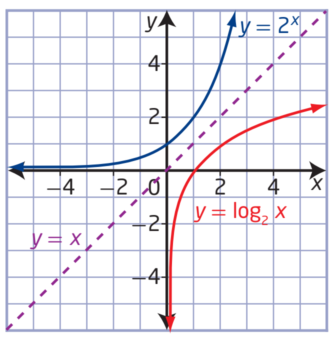
The largest measured earthquake struck Chile in 1960. It measured 9.5 on the Richter scale. How many times as great was the seismic shaking of the Chilean earthquake than the 1949 Haida Gwaii earthquake, which measured 8.1 on the Richter scale?

Key Ideas

- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form	**Logarithmic Form**
$x = c^y$	$y = \log_c x$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.



- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in \mathrm{R}\}$
  - the range is $\{y \mid y \in \mathrm{R}\}$
  - the x-intercept is 1
  - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

  $\log_{10} x = \log x$

Check Your Understanding

Practise

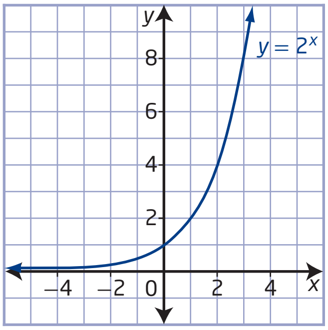
1. For each exponential graph,

i) copy the graph on grid paper, and then sketch the graph of the inverse on the same grid

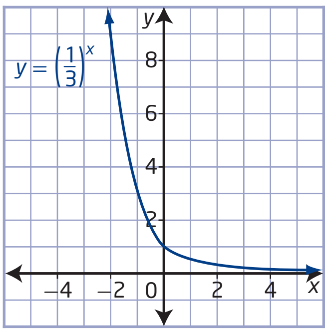
ii) write the equation of the inverse

iii) determine the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equation of the asymptote

a)



b)



2. Express in logarithmic form.

a) $12^2 = 144$ **b)** $8^{\frac{1}{3}} = 2$

c) $10^{-5} = 0.000\ 01$ **d)** $7^{2x} = y + 3$

3. Express in exponential form.

a) $\log_5 25 = 2$

b) $\log_8 4 = \frac{2}{3}$

c) $\log 1\ 000\ 000 = 6$

d) $\log_{11} (x + 3) = y$

4. Use the definition of a logarithm to evaluate.

a) $\log_5 125$

b) $\log 1$

c) $\log_4 \sqrt[3]{4}$

d) $\log_{\frac{1}{3}} 27$

5. Without using technology, find two consecutive whole numbers, a and b, such that $a < \log_2 28 < b$.

6. State a value of x so that $\log_3 x$ is

a) a positive integer

b) a negative integer

c) zero

d) a rational number

7. The base of a logarithm can be any positive real number except 1. Use examples to illustrate why the base of a logarithm cannot be

a) 0

b) 1

c) negative

8. a) If $f(x) = 5^x$, state the equation of the inverse, $f^{-1}(x)$.

b) Sketch the graph of $f(x)$ and its inverse. Identify the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

9. a) If $g(x) = \log_{\frac{1}{4}} x$, state the equation of the inverse, $g^{-1}(x)$.

b) Sketch the graph of $g(x)$ and its inverse. Identify the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

Apply

10. Explain the relationship between the characteristics of the functions $y = 7^x$ and $y = \log_7 x$.

11. Graph $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ on the same coordinate grid. Describe the ways the graphs are

a) alike

b) different

12. Determine the value of x in each.

a) $\log_6 x = 3$

b) $\log_x 9 = \frac{1}{2}$

c) $\log_{\frac{1}{4}} x = -3$

d) $\log_x 16 = \frac{4}{3}$

13. Evaluate each expression.

a) 5^m, where $m = \log_5 7$

b) 8^n, where $n = \log_8 6$

14. Evaluate.

a) $\log_2 (\log_3 (\log_4 64))$

b) $\log_4 (\log_2 (\log 10^{16}))$

15. Determine the x-intercept of $y = \log_7 (x + 2)$.

16. The point $\left(\frac{1}{8}, -3\right)$ is on the graph of the logarithmic function $f(x) = \log_c x$, and the point $(4, k)$ is on the graph of the inverse, $y = f^{-1}(x)$. Determine the value of k.

17. The growth of a new social networking site can be modelled by the exponential function $N(t) = 1.1^t$, where N is the number of users after t days.

a) Write the equation of the inverse.

b) How long will it take, to the nearest day, for the number of users to exceed 1 000 000?

18. The Palermo Technical Impact Hazard scale was developed to rate the potential hazard impact of a near-Earth object. The Palermo scale, P, is defined as $P = \log R$, where R is the relative risk. Compare the relative risks of two asteroids, one with a Palermo scale value of -1.66 and the other with a Palermo scale value of -4.83.

19. The formula for the Richter magnitude, M, of an earthquake is $M = \log \frac{A}{A_0}$, where A is the amplitude of the ground motion and A_0 is the amplitude of a standard earthquake. In 1985, an earthquake with magnitude 6.9 on the Richter scale was recorded in the Nahanni region of the Northwest Territories. The largest recorded earthquake in Saskatchewan occurred in 1982 near the town of Big Beaver. It had a magnitude of 3.9 on the Richter scale. How many times as great as the seismic shaking of the Saskatchewan earthquake was that of the Nahanni earthquake?

Did You Know?

Scientists at the Geological Survey of Canada office, near Sidney, British Columbia, record and locate earthquakes every day. Of the approximately 1000 earthquakes each year in Western Canada, fewer than 50 are strong enough to be felt by humans. In Canada, there have been no casualties directly related to earthquakes. A tsunami triggered by a major earthquake off the coast of California could be hazardous to the British Columbia coast.

20. If $\log_5 x = 2$, then determine $\log_5 125x$.

Extend

21. If $\log_3 (m - n) = 0$ and $\log_3 (m + n) = 3$, determine the values of m and n.

22. If $\log_3 m = n$, then determine $\log_3 m^4$, in terms of n.

23. Determine the equation of the inverse of $y = \log_2 (\log_3 x)$.

24. If $m = \log_2 n$ and $2m + 1 = \log_2 16n$, determine the values of m and n.

Create Connections

C1 Graph $y = |\log_2 x|$. Describe how the graph of $y = |\log_2 x|$ is related to the graph of $y = \log_2 x$.

C2 Create a mind map to summarize everything you know about the graph of the logarithmic function $y = \log_c x$, where $c > 0$ and $c \neq 1$. Enhance your mind map by sharing ideas with classmates.

C3 **MINI LAB** Recall that an irrational number cannot be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Irrational numbers cannot be expressed as a terminating or a repeating decimal. The number π is irrational. Another special irrational number is represented by the letter e. Its existence was implied by John Napier, the inventor of logarithms, but it was later studied by the Swiss mathematician Leonhard Euler. Euler was the first to use the letter e to represent it, and, as a result, e is sometimes called Euler's number.

Step 1 The number e can be approximated in a variety of ways.

a) Use the e or e^x key on a calculator to find the decimal approximation of e to nine decimal places.

b) You can obtain a better approximation of the number e by substituting larger values for x in the expression $\left(1 + \frac{1}{x}\right)^x$.

x	$\left(1 + \frac{1}{x}\right)^x$
10	2.593 742 460
100	2.704 813 829
1 000	2.716 923 932
10 000	2.718 145 927

As a power of 10, what is the minimum value of x needed to approximate e correctly to nine decimal places?

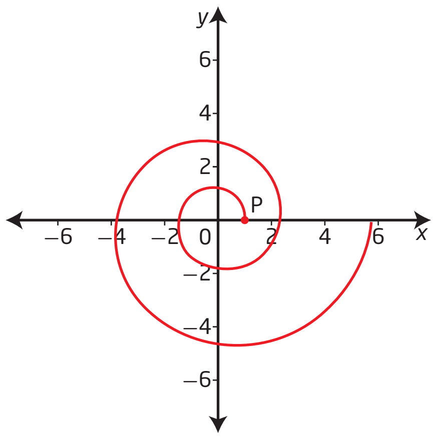
Step 2 **a)** Graph the inverse of the exponential function $y = e^x$. Identify the following characteristics of the inverse graph:

- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equation of the asymptote

b) A logarithm to base e is called a *natural logarithm*. The natural logarithm of any positive real number x is denoted by $\log_e x$ or $\ln x$. What is the inverse of the exponential function $y = e^x$?

Step 3 The shell of the chambered nautilus is a logarithmic spiral. Other real-world examples of logarithmic spirals are the horns of wild sheep, the curve of elephant tusks, the approach of a hawk to its prey, and the arms of spiral galaxies.

A logarithmic spiral can be formed by starting at point P(1, 0) and then rotating point P counterclockwise an angle of θ, in radians, such that the distance, r, from point P to the origin is always $r = e^{0.14\theta}$.



a) Determine the distance, r, from point P to the origin after the point has rotated 2π. Round your answer to two decimal places.

b) The spiral is logarithmic because the relationship between r and θ may be expressed using logarithms.

i) Express $r = e^{0.14\theta}$ in logarithmic form.

ii) Determine the angle, θ, of rotation that corresponds to a value for r of 12. Give your answer in radians to two decimal places.

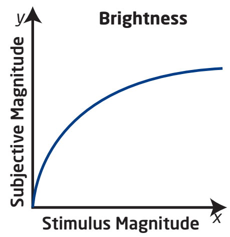
8.2

Transformations of Logarithmic Functions

Focus on…

- explaining the effects of the parameters a, b, h, and k in $y = a \log_c (b(x - h)) + k$ on the graph of $y = \log_c x$, where $c > 1$
- sketching the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_c x$, where $c > 1$, and stating the characteristics of the graph

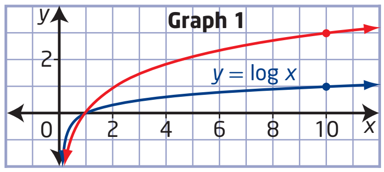
In some situations people are less sensitive to differences in the magnitude of a stimulus as the intensity of the stimulus increases. For example, if you compare a 50-W light bulb to a 100-W light bulb, the 100-W light bulb seems much brighter. However, if you compare a 150-W light bulb to a 200-W light bulb, they appear almost the same. In 1860, Gustav Fechner, the founder of psychophysics, proposed a logarithmic curve to describe this relationship.

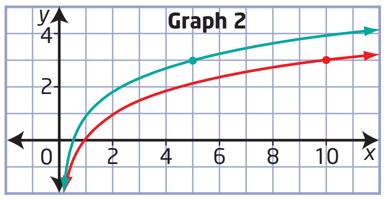


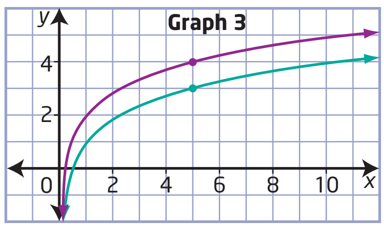
Describe another situation that might be modelled by a logarithmic curve.

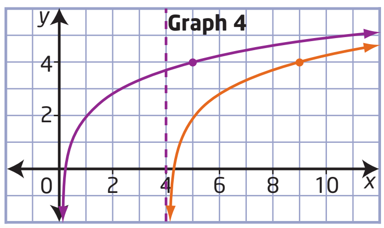
Investigate Transformations of Logarithmic Functions

1. The graphs show how $y = \log x$ is transformed into $y = a \log (b(x - h)) + k$ by changing one parameter at a time. Graph 1 shows $y = \log x$ and the effect of changing one parameter. The effect on one key point is shown at each step. For graphs 1 to 4, describe the effect of the parameter introduced and write the equation of the transformed function.









2. Suppose that before the first transformation, $y = \log x$ is reflected in an axis. Describe the effect on the equation if the reflection is in

a) the x-axis

b) the y-axis

Reflect and Respond

3. In general, describe how the parameters a, b, h, and k in the logarithmic function $y = a \log_c (b(x - h)) + k$ affect the following characteristics of $y = \log_c x$.

a) the domain

b) the range

c) the vertical asymptote

Link the Ideas

The graph of the logarithmic function $y = a \log_c (b(x - h)) + k$ can be obtained by transforming the graph of $y = \log_c x$. The table below uses mapping notation to show how each parameter affects the point (x, y) on the graph of $y = \log_c x$.

Parameter	Transformation
a	$(x, y) \rightarrow (x, ay)$
b	$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$
h	$(x, y) \rightarrow (x + h, y)$
k	$(x, y) \rightarrow (x, y + k)$

How would you describe the effects of each parameter?

Example 1

Translations of a Logarithmic Function

a) Use transformations to sketch the graph of the function $y = \log_3 (x + 9) + 2$.

b) Identify the following characteristics of the graph of the function.

i) the equation of the asymptote
ii) the domain and range
iii) the y-intercept, if it exists
iv) the x-intercept, if it exists

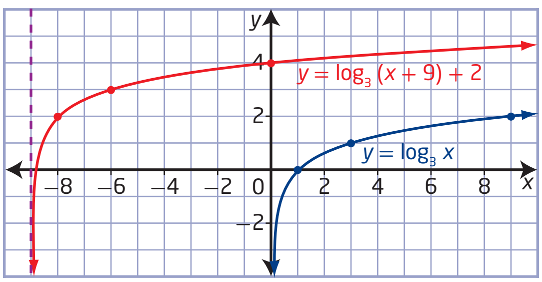
Solution

a) To sketch the graph of $y = \log_3 (x + 9) + 2$, translate the graph of $y = \log_3 x$ to the left 9 units and up 2 units.

Choose some key points to sketch the base function, $y = \log_3 x$. Examine how the coordinates of key points and the position of the asymptote change. Each point (x, y) on the graph of $y = \log_3 x$ is translated to become the point $(x - 9, y + 2)$ on the graph of $y = \log_3 (x + 9) + 2$.

In mapping notation, $(x, y) \rightarrow (x - 9, y + 2)$.

$y = \log_3 x$	$y = \log_3 (x + 9) + 2$
(1, 0)	(−8, 2)
(3, 1)	(−6, 3)
(9, 2)	(0, 4)



b) **i)** For $y = \log_3 x$, the asymptote is the y-axis, that is, the equation $x = 0$. For $y = \log_3 (x + 9) + 2$, the equation of the asymptote occurs when $x + 9 = 0$. Therefore, the equation of the vertical asymptote is $x = -9$.

ii) The domain is $\{x \mid x > -9, x \in \mathrm{R}\}$ and the range is $\{y \mid y \in \mathrm{R}\}$.

iii) To determine the y-intercept, substitute $x = 0$. Then, solve for y.

$$y = \log_3 (0 + 9) + 2$$
$$y = \log_3 9 + 2$$
$$y = 2 + 2$$
$$y = 4$$

The y-intercept is 4.

iv) To determine the x-intercept, substitute $y = 0$. Then, solve for x.

$$\begin{aligned} 0 &= \log_3 (x + 9) + 2 \\ -2 &= \log_3 (x + 9) \\ 3^{-2} &= x + 9 \\ \frac{1}{9} &= x + 9 \\ -\frac{80}{9} &= x \end{aligned}$$

The x-intercept is $-\frac{80}{9}$ or approximately -8.9.

Your Turn

a) Use transformations to sketch the graph of the function $y = \log (x - 10) - 1$.

b) Identify the following characteristics of the graph of the function.

i) the equation of the asymptote

ii) the domain and range

iii) the y-intercept, if it exists

iv) the x-intercept, if it exists

Example 2

Reflections, Stretches, and Translations of a Logarithmic Function

a) Use transformations to sketch the graph of the function $y = -\log_2 (2x + 6)$.

b) Identify the following characteristics of the graph of the function.

i) the equation of the asymptote

ii) the domain and range

iii) the y-intercept, if it exists

iv) the x-intercept, if it exists

Solution

a) Factor the expression $2x + 6$ to identify the horizontal translation.

$$y = -\log_2 (2x + 6)$$
$$y = -\log_2 (2(x + 3))$$

To sketch the graph of $y = -\log_2 (2(x + 3))$ from the graph of $y = \log_2 x$,

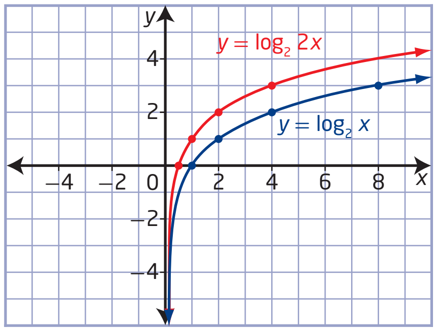
- horizontally stretch about the y-axis by a factor of $\frac{1}{2}$
- reflect in the x-axis
- horizontally translate 3 units to the left

Start by horizontally stretching about the y-axis by a factor of $\frac{1}{2}$.

Key points on the graph of $y = \log_2 x$ change as shown.

In mapping notation, $(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$.

$y = \log_2 x$	$y = \log_2 2x$
(1, 0)	(0.5, 0)
(2, 1)	(1, 1)
(4, 2)	(2, 2)
(8, 3)	(4, 3)

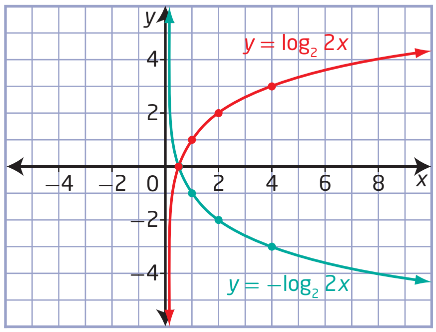


At this stage, the asymptote remains unchanged: it is the vertical line $x = 0$.

Next, reflect in the x-axis. The key points change as shown.

In mapping notation, $(x, y) \rightarrow (x, -y)$.

$y = \log_2 2x$	$y = -\log_2 2x$
(0.5, 0)	(0.5, 0)
(1, 1)	(1, −1)
(2, 2)	(2, −2)
(4, 3)	(4, −3)

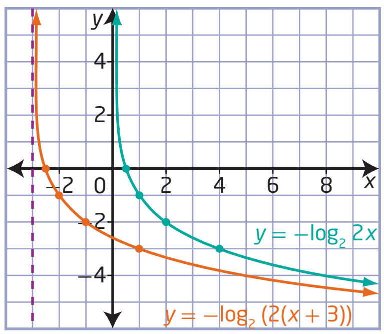


The asymptote is still $x = 0$ at this stage.

Lastly, translate horizontally 3 units to the left. The key points change as shown.

In mapping notation, $(x, y) \rightarrow (x - 3, y)$.

$y = -\log_2 2x$	$y = -\log_2 (2(x + 3))$
(0.5, 0)	(−2.5, 0)
(1, −1)	(−2, −1)
(2, −2)	(−1, −2)
(4, −3)	(1, −3)



The asymptote is now shifted 3 units to the left to become the vertical line $x = -3$.

b) **i)** From the equation of the function, $y = -\log_2 (2x + 6)$, the equation of the vertical asymptote occurs when $2x + 6 = 0$. Therefore, the equation of the vertical asymptote is $x = -3$.

ii) The domain is $\{x \mid x > -3, x \in \mathrm{R}\}$ and the range is $\{y \mid y \in \mathrm{R}\}$.

iii) To determine the y-intercept from the equation of the function, substitute $x = 0$. Then, solve for y.

$y = -\log_2 (2(0) + 6)$

$y = -\log_2 6$ Use a calculator to determine the approximate value.

$y \approx -2.6$

The y-intercept is approximately -2.6.

iv) To determine the x-intercept from the equation of the function, substitute $y = 0$. Then, solve for x.

$$\begin{aligned} 0 &= -\log_2 (2x + 6) \\ 0 &= \log_2 (2x + 6) \\ 2^0 &= 2x + 6 \\ 1 &= 2x + 6 \\ -5 &= 2x \\ -\frac{5}{2} &= x \end{aligned}$$

The x-intercept is $-\frac{5}{2}$ or -2.5.

Your Turn

a) Use transformations to sketch the graph of the function $y = 2 \log_3 (-x + 1)$.

b) Identify the following characteristics.

i) the equation of the asymptote

ii) the domain and range

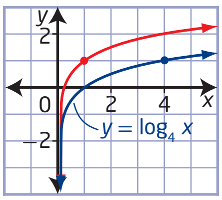
iii) the y-intercept, if it exists

iv) the x-intercept, if it exists

Example 3

Determine the Equation of a Logarithmic Function Given Its Graph

The red graph can be generated by stretching the blue graph of $y = \log_4 x$. Write the equation that describes the red graph.



Solution

The red graph has been horizontally stretched since a vertical stretch does not change the x-intercept.

Method 1: Compare With the Graph of $y = \log_4 x$

The key point $(4, 1)$ on the graph of $y = \log_4 x$ has become the image point $(1, 1)$ on the red graph. Thus, the red graph can be generated by horizontally stretching the graph of $y = \log_4 x$ about the y-axis by a factor of $\frac{1}{4}$. The red graph can be described by the equation $y = \log_4 4x$.

Method 2: Use Points and Substitution

The equation of the red graph is of the form $y = \log_4 bx$. Substitute the coordinates of a point on the red graph, such as (4, 2), into the equation. Solve for b.

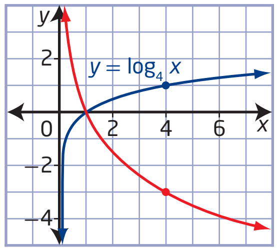
$$y = \log_4 bx$$
$$2 = \log_4 4b$$
$$4^2 = 4b$$
$$4 = b$$

Which other point could you have used?

The red graph can be described by the equation $y = \log_4 4x$.

Your Turn

The red graph can be generated by stretching and reflecting the graph of $y = \log_4 x$. Write the equation that describes the red graph.



Example 4

An Application Involving a Logarithmic Function

Welding is the most common way to permanently join metal parts together. Welders wear helmets fitted with a filter shade to protect their eyes from the intense light and radiation produced by a welding light. The filter shade number, N, is defined by the function $N = \frac{7(-\log T)}{3} + 1$, where T is the fraction of visible light that passes through the filter. Shade numbers range from 2 to 14, with a lens shade number of 14 allowing the least amount of light to pass through.

The correct filter shade depends on the type of welding. A shade number 12 is suggested for arc welding. What fraction of visible light is passed through the filter to the welder, as a percent to the nearest ten thousandth?

Solution

Substitute 12 for N and solve for T.

$$12 = -\frac{7}{3}\log_{10} T + 1$$
$$11 = -\frac{7}{3}\log_{10} T$$
$$11\left(-\frac{3}{7}\right) = \log_{10} T$$
$$-\frac{33}{7} = \log_{10} T$$
$$10^{-\frac{33}{7}} = T$$
$$0.000\,019 \approx T$$

A filter shade number 12 allows approximately 0.000 019, or 0.0019%, of the visible light to pass through the filter.

Your Turn

There is a logarithmic relationship between butterflies and flowers. In one study, scientists found that the relationship between the number, F, of flower species that a butterfly feeds on and the number, B, of butterflies observed can be modelled by the function $F = -2.641 + 8.958 \log B$.

Predict the number of butterfly observations in a region with 25 flower species.

Arctic butterfly, oeneis chryxus

Did You Know?

Eighty-seven different species of butterfly have been seen in Nunavut. Northern butterflies survive the winters in a larval stage and manufacture their own antifreeze to keep from freezing. They manage the cool summer temperatures by angling their wings to catch the sun's rays.

Key Ideas

- To represent real-life situations, you may need to transform the basic logarithmic function $y = \log_b x$ by applying reflections, stretches, and translations. These transformations should be performed in the same manner as those applied to any other function.
- The effects of the parameters a, b, h, and k in $y = a \log_c (b(x - h)) + k$ on the graph of the logarithmic function $y = \log_c x$ are shown below.

Vertically stretch by a factor of $|a|$ about the x-axis. Reflect in the x-axis if $a < 0$.

Vertically translate k units.

$$y = a \log_c (b(x - h)) + k$$

Horizontally stretch by a factor of $\left|\frac{1}{b}\right|$ about the y-axis. Reflect in the y-axis if $b < 0$.

Horizontally translate h units.

- Only parameter h changes the vertical asymptote and the domain. None of the parameters change the range.

Check Your Understanding

Practise

1. Describe how the graph of each logarithmic function can be obtained from the graph of $y = \log_5 x$.
   a) $y = \log_5 (x - 1) + 6$
   b) $y = -4 \log_5 3x$
   c) $y = \frac{1}{2} \log_5 (-x) + 7$

2. **a)** Sketch the graph of $y = \log_3 x$, and then apply, in order, each of the following transformations.
   - Stretch vertically by a factor of 2 about the x-axis.
   - Translate 3 units to the left.

   b) Write the equation of the final transformed image.

3. a) Sketch the graph of $y = \log_2 x$, and then apply, in order, each of the following transformations.
   - Reflect in the y-axis.
   - Translate vertically 5 units up.

   b) Write the equation of the final transformed image.

4. Sketch the graph of each function.

   a) $y = \log_2 (x + 4) - 3$

   b) $y = -\log_3 (x + 1) + 2$

   c) $y = \log_4 (-2(x - 8))$

5. Identify the following characteristics of the graph of each function.

   i) the equation of the asymptote

   ii) the domain and range

   iii) the y-intercept, to one decimal place if necessary

   iv) the x-intercept, to one decimal place if necessary

   a) $y = -5 \log_3 (x + 3)$

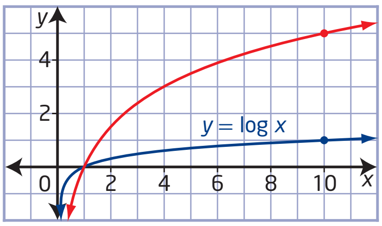
   b) $y = \log_6 (4(x + 9))$

   c) $y = \log_5 (x + 3) - 2$

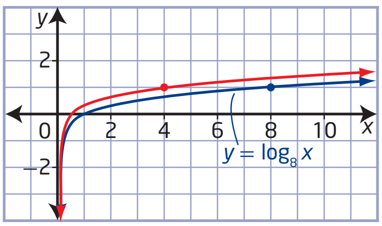
   d) $y = -3 \log_2 (x + 1) - 6$

6. In each, the red graph is a stretch of the blue graph. Write the equation of each red graph.

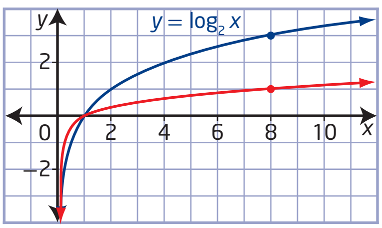
   a)



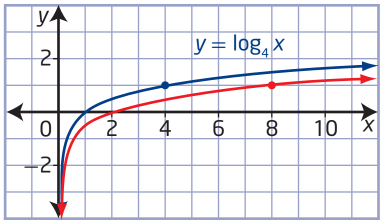
   b)



   c)



   d)



7. Describe, in order, a series of transformations that could be applied to the graph of $y = \log_7 x$ to obtain the graph of each function.

   a) $y = \log_7 (4(x + 5)) + 6$

   b) $y = 2 \log_7 \left(-\frac{1}{3}(x - 1)\right) - 4$

Apply

8. The graph of $y = \log_3 x$ has been transformed to $y = a \log_3 (b(x - h)) + k$. Find the values of a, b, h, and k for each set of transformations. Write the equation of the transformed function.

   a) a reflection in the x-axis and a translation of 6 units left and 3 units up

   b) a vertical stretch by a factor of 5 about the x-axis and a horizontal stretch about the y-axis by a factor of $\frac{1}{3}$

   c) a vertical stretch about the x-axis by a factor of $\frac{3}{4}$, a horizontal stretch about the y-axis by a factor of 4, a reflection in the y-axis, and a translation of 2 units right and 5 units down

9. Describe how the graph of each logarithmic function could be obtained from the graph of $y = \log_3 x$.

   a) $y = 5 \log_3 (-4x + 12) - 2$

   b) $y = -\frac{1}{4} \log_3 (6 - x) + 1$

10. a) Only a vertical translation has been applied to the graph of $y = \log_3 x$ so that the graph of the transformed image passes through the point $(9, -4)$. Determine the equation of the transformed image.

    b) Only a horizontal stretch has been applied to the graph of $y = \log_2 x$ so that the graph of the transformed image passes through the point $(8, 1)$. Determine the equation of the transformed image.

11. Explain how the graph of $\frac{1}{3}(y + 2) = \log_6 (x - 4)$ can be generated by transforming the graph of $y = \log_6 x$.

12. The equivalent amount of energy, E, in kilowatt-hours (kWh), released for an earthquake with a Richter magnitude of R is determined by the function $R = 0.67 \log 0.36E + 1.46$.

a) Describe how the function is transformed from $R = \log E$.

b) The strongest earthquake in Eastern Canada occurred in 1963 at Charlevoix, Québec. It had a Richter magnitude of 7.0. What was the equivalent amount of energy released, to the nearest kilowatt-hour?

13. In a study, doctors found that in young people the arterial blood pressure, P, in millimetres of mercury (mmHg), is related to the vessel volume, V, in microlitres (µL), of the radial artery by the logarithmic function $V = 0.23 + 0.35 \log (P - 56.1)$, $P > 56.1$.

a) To the nearest tenth of a microlitre, predict the vessel volume when the arterial blood pressure is 110 mmHg.

b) To the nearest millimetre of mercury, predict the arterial blood pressure when the vessel volume is 0.7 µL.

14. According to the Ehrenberg relation, the average measurements of heights, h, in centimetres, and masses, m, in kilograms, of children between the ages of 5 and 13 are related by the function $\log m = 0.008h + 0.4$.

a) Predict the height of a 10-year-old child with a mass of 60 kg, to the nearest centimetre.

b) Predict the mass of a 12-year-old child with a height of 150 cm, to the nearest kilogram.

Extend

15. The graph of $f(x) = \log_8 x$ can also be described by the equation $g(x) = a \log_2 x$. Find the value of a.

16. Determine the equation of the transformed image after the transformations described are applied to the given graph.

a) The graph of $y = 2 \log_5 x - 7$ is reflected in the x-axis and translated 6 units up.

b) The graph of $y = \log (6(x - 3))$ is stretched horizontally about the y-axis by a factor of 3 and translated 9 units left.

17. The graph of $f(x) = \log_2 x$ has been transformed to $g(x) = a \log_2 x + k$. The transformed image passes through the points $\left(\frac{1}{4}, -9\right)$ and $(16, -6)$. Determine the values of a and k.

Create Connections

C1 The graph of $f(x) = 5^x$ is

- reflected in the line $y = x$
- vertically stretched about the x-axis by a factor of $\frac{1}{4}$
- horizontally stretched about the y-axis by a factor of 3
- translated 4 units right and 1 unit down

If the equation of the transformed image is written in the form $g(x) = a \log_c (b(x - h)) + k$, determine the values of a, b, h, and k. Write the equation of the function $g(x)$.

C2 **a)** Given $f(x) = \log_2 x$, write the equations for the functions $y = -f(x)$, $y = f(-x)$, and $y = f^{-1}(x)$.

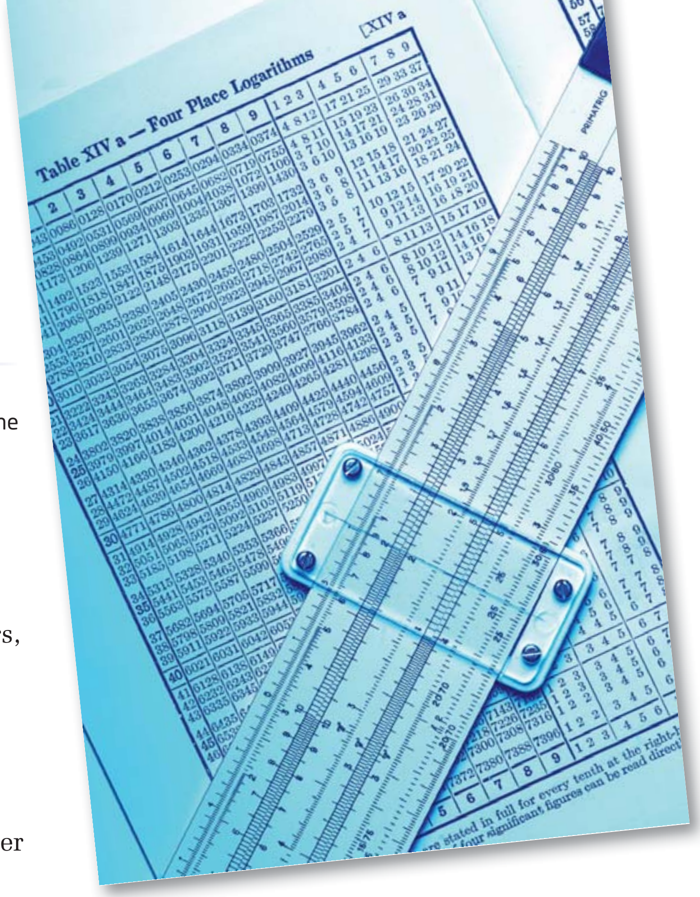
b) Sketch the graphs of the four functions in part a). Describe how each transformed graph can be obtained from the graph of $f(x) = \log_2 x$.

C3 **a)** The graph of $y = 3(7^{2x - 1}) + 5$ is reflected in the line $y = x$. What is the equation of the transformed image?

b) If $f(x) = 2 \log_3 (x - 1) + 8$, find the equation of $f^{-1} (x)$.

C4 Create a poster, digital presentation, or video to illustrate the different transformations you studied in this section.

8.3 Laws of Logarithms



Focus on…

- developing the laws of logarithms
- determining an equivalent form of a logarithmic expression using the laws of logarithms
- applying the laws of logarithms to logarithmic scales

Today you probably take hand-held calculators for granted. But John Napier, the inventor of logarithms, lived in a time when scientists, especially astronomers, spent much time performing tedious arithmetic calculations on paper. Logarithms revolutionized mathematics and science by simplifying these calculations. Using the laws of logarithms, you can convert multiplication to addition and division to subtraction. The French mathematician and astronomer Pierre-Simon Laplace claimed that logarithms, "by shortening the labours, doubled the life of the astronomer." This allowed scientists to be more productive. Many of the advances in science would not have been possible without the invention of logarithms.

The laws that made logarithms so useful as a calculation tool are still important. They can be used to simplify logarithmic functions and expressions and in solving both exponential and logarithmic equations.

Did You Know?

The world's first hand-held scientific calculator was the Hewlett-Packard HP-35, so called because it had 35 keys. Introduced in 1972, it retailed for approximately U.S. $395. Market research at the time warned that the demand for a pocket-sized calculator was too small. Hewlett-Packard estimated that they needed to sell 10 000 calculators in the first year to break even. They ended up selling 10 times that. By the time it was discontinued in 1975, sales of the HP-35 exceeded 300 000.

Investigate the Laws of Logarithms

1. a) Show that $\log(1000 \times 100) \neq (\log 1000)(\log 100)$.

b) Use a calculator to find the approximate value of each expression, to four decimal places.

i) $\log 6 + \log 5$ **ii)** $\log 21$

iii) $\log 11 + \log 9$ **iv)** $\log 99$

v) $\log 7 + \log 3$ **vi)** $\log 30$

c) Based on the results in part b), suggest a possible law for $\log M + \log N$, where M and N are positive real numbers.

d) Use your conjecture from part c) to express $\log 1000 + \log 100$ as a single logarithm.

2. a) Show that $\log \frac{1000}{100} \neq \frac{\log 1000}{\log 100}$.

b) Use a calculator to find the approximate value of each expression, to four decimal places.

i) $\log 12$	**ii)** $\log 35 - \log 5$
iii) $\log 36$	**iv)** $\log 72 - \log 2$
v) $\log 48 - \log 4$	**vi)** $\log 7$

c) Based on the results in part b), suggest a possible law for $\log M - \log N$, where M and N are positive real numbers.

d) Use your conjecture from part c) to express $\log 1000 - \log 100$ as a single logarithm.

3. a) Show that $\log 1000^2 \neq (\log 1000)^2$.

b) Use a calculator to find the approximate value of each expression, to four decimal places.

i) $3 \log 5$	**ii)** $\log 49$
iii) $\log 125$	**iv)** $\log 16$
v) $4 \log 2$	**vi)** $2 \log 7$

c) Based on the results in part b), suggest a possible law for $P \log M$, where M is a positive real number and P is any real number.

d) Use your conjecture from part c) to express $2 \log 1000$ as a logarithm without a coefficient.

Reflect and Respond

4. The laws of common logarithms are also true for any logarithm with a base that is a positive real number other than 1. Without using technology, evaluate each of the following.

a) $\log_6 18 + \log_6 2$

b) $\log_2 40 - \log_2 5$

c) $4 \log_9 3$

5. Each of the three laws of logarithms corresponds to one of the three laws of powers:

- product law of powers: $(c^x)(c^y) = c^{x+y}$
- quotient law of powers: $\frac{c^x}{c^y} = c^{x-y}$, $c \neq 0$
- power of a power law: $(c^x)^y = c^{xy}$

Explain how the laws of logarithms are related to the laws of powers.

Link the Ideas

Since logarithms are exponents, the laws of logarithms are related to the laws of powers.

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$\log_c MN = \log_c M + \log_c N$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$MN = (c^x)(c^y)$

$MN = c^{x+y}$ Apply the product law of powers.

$\log_c MN = x + y$ Write in logarithmic form.

$\log_c MN = \log_c M + \log_c N$ Substitute for x and y.

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$\log_c \frac{M}{N} = \log_c M - \log_c N$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$\frac{M}{N} = \frac{c^x}{c^y}$

$\frac{M}{N} = c^{x-y}$ Apply the quotient law of powers.

$\log_c \frac{M}{N} = x - y$ Write in logarithmic form.

$\log_c \frac{M}{N} = \log_c M - \log_c N$ Substitute for x and y.

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$\log_c M^P = P \log_c M$

How could you prove the quotient law using the product law and the power law?

Proof

Let $\log_c M = x$, where M and c are positive real numbers with $c \neq 1$.

Write the equation in exponential form as $M = c^x$.

Let P be a real number.

$$
\begin{aligned}
M &= c^x \\
M^P &= (c^x)^P \\
M^P &= c^{xP} && \text{Simplify the exponents.} \\
\log_c M^P &= xP && \text{Write in logarithmic form.} \\
\log_c M^P &= (\log_c M)P && \text{Substitute for } x. \\
\log_c M^P &= P \log_c M
\end{aligned}
$$

The laws of logarithms can be applied to logarithmic functions, expressions, and equations.

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_5 \dfrac{xy}{z}$

b) $\log_7 \sqrt[3]{x}$

c) $\log_6 \dfrac{1}{x^2}$

d) $\log \dfrac{x^3}{y\sqrt{z}}$

Solution

a) $$\begin{aligned}\log_5 \frac{xy}{z} &= \log_5 xy - \log_5 z \\ &= \log_5 x + \log_5 y - \log_5 z\end{aligned}$$

b) $$\begin{aligned}\log_7 \sqrt[3]{x} &= \log_7 x^{\frac{1}{3}} \\ &= \frac{1}{3} \log_7 x\end{aligned}$$

c) $$\begin{aligned}\log_6 \frac{1}{x^2} &= \log_6 x^{-2} \\ &= -2 \log_6 x\end{aligned}$$

You could also start by applying the quotient law to the original expression. Try this. You should arrive at the same answer.

d) $$\begin{aligned}\log \frac{x^3}{y\sqrt{z}} &= \log x^3 - \log y\sqrt{z} \\ &= \log x^3 - \left(\log y + \log z^{\frac{1}{2}}\right) \\ &= 3 \log x - \log y - \frac{1}{2} \log z\end{aligned}$$

Your Turn

Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_6 \dfrac{x}{y}$

b) $\log_5 \sqrt{xy}$

c) $\log_3 \dfrac{9}{\sqrt[3]{x^2}}$

d) $\log_7 \dfrac{x^5 y}{\sqrt{z}}$

Example 2

Use the Laws of Logarithms to Evaluate Expressions

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_6 8 + \log_6 9 - \log_6 2$

b) $\log_7 7\sqrt{7}$

c) $2 \log_2 12 - \left(\log_2 6 + \frac{1}{3} \log_2 27\right)$

Solution

a)
$$\begin{aligned} &\log_6 8 + \log_6 9 - \log_6 2 \\ &= \log_6 \frac{8 \times 9}{2} \\ &= \log_6 36 \\ &= \log_6 6^2 \\ &= 2 \end{aligned}$$

b)
$$\begin{aligned} &\log_7 7\sqrt{7} \\ &= \log_7 \left(7 \times 7^{\frac{1}{2}}\right) \\ &= \log_7 7 + \log_7 7^{\frac{1}{2}} \\ &= \log_7 7 + \frac{1}{2} \log_7 7 \\ &= 1 + \frac{1}{2}(1) \\ &= \frac{3}{2} \end{aligned}$$

How can you use your knowledge of exponents to evaluate this expression using only the power law for logarithms?

c)
$$\begin{aligned} &2 \log_2 12 - \left(\log_2 6 + \frac{1}{3} \log_2 27\right) \\ &= \log_2 12^2 - \left(\log_2 6 + \log_2 27^{\frac{1}{3}}\right) \\ &= \log_2 144 - (\log_2 6 + \log_2 \sqrt[3]{27}) \\ &= \log_2 144 - (\log_2 6 + \log_2 3) \\ &= \log_2 144 - \log_2 (6 \times 3) \\ &= \log_2 \frac{144}{18} \\ &= \log_2 8 \\ &= 3 \end{aligned}$$

Your Turn

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_3 9\sqrt{3}$

b) $\log_5 1000 - \log_5 4 - \log_5 2$

c) $2 \log_3 6 - \frac{1}{2} \log_3 64 + \log_3 2$

Example 3

Use the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $\log_7 x^2 + \log_7 x - \dfrac{5 \log_7 x}{2}$

b) $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3)$

Solution

a)
$$\begin{aligned}
&\log_7 x^2 + \log_7 x - \frac{5 \log_7 x}{2}\\
&= \log_7 x^2 + \log_7 x - \frac{5}{2} \log_7 x\\
&= \log_7 x^2 + \log_7 x - \log_7 x^{\frac{5}{2}}\\
&= \log_7 \frac{(x^2)(x)}{x^{\frac{5}{2}}}\\
&= \log_7 x^{2 + 1 - \frac{5}{2}}\\
&= \log_7 x^{\frac{1}{2}}\\
&= \frac{1}{2} \log_7 x,\ x > 0
\end{aligned}$$

The logarithmic expression is written as a single logarithm that cannot be further simplified by the laws of logarithms.

b)
$$\begin{aligned}
&\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3)\\
&= \log_5 \frac{2x - 2}{x^2 + 2x - 3}\\
&= \log_5 \frac{2\cancel{(x - 1)}^{1}}{(x + 3)\cancel{(x - 1)}_{1}}\\
&= \log_5 \frac{2}{x + 3}
\end{aligned}$$

For the original expression to be defined, both logarithmic terms must be defined.

$$\begin{aligned}
2x - 2 &> 0 & \qquad x^2 + 2x - 3 &> 0\\
2x &> 2 & (x + 3)(x - 1) &> 0\\
x &> 1 \quad \text{and} & x < -3 \text{ or } x &> 1
\end{aligned}$$

What other methods could you have used to solve this quadratic inequality?

The conditions $x > 1$ and $x < -3$ or $x > 1$ are both satisfied when $x > 1$.

Hence, the variable x needs to be restricted to $x > 1$ for the original expression to be defined and then written as a single logarithm.

Therefore, $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3) = \log_5 \dfrac{2}{x + 3}$, $x > 1$.

Your Turn

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $4 \log_3 x - \dfrac{1}{2}(\log_3 x + 5 \log_3 x)$

b) $\log_2 (x^2 - 9) - \log_2 (x^2 - x - 6)$

Example 4

Solve a Problem Involving a Logarithmic Scale

The human ear is sensitive to a large range of sound intensities. Scientists have found that the sensation of loudness can be described using a logarithmic scale. The intensity level, β, in decibels, of a sound is defined as $\beta = 10 \log \frac{I}{I_0}$, where I is the intensity of the sound, in watts per square metre (W/m^2), and I_0 is 10^{-12} W/m^2, corresponding to the faintest sound that can be heard by a person of normal hearing.

a) Audiologists recommend that people should wear hearing protection if the sound level exceeds 85 dB. The sound level of a chainsaw is about 85 dB. The maximum volume setting of a portable media player with headphones is about 110 dB. How many times as intense as the sound of the chainsaw is the maximum volume setting of the portable media player?

b) Sounds that are at most 100 000 times as intense as a whisper are considered safe, no matter how long or how often you hear them. The sound level of a whisper is about 20 dB. What sound level, in decibels, is considered safe no matter how long it lasts?

Solution

a) Let the decibel levels of two sounds be $\beta_1 = 10 \log \frac{I_1}{I_0}$ and $\beta_2 = 10 \log \frac{I_2}{I_0}$.

Then, compare the two intensities.

$$\beta_2 - \beta_1 = 10 \log \frac{I_2}{I_0} - 10 \log \frac{I_1}{I_0}$$

$$\beta_2 - \beta_1 = 10\left(\log \frac{I_2}{I_0} - \log \frac{I_1}{I_0}\right)$$

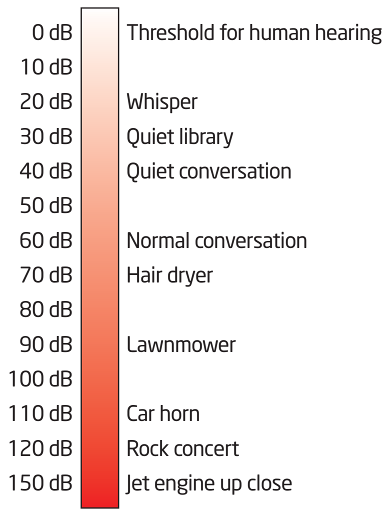
$$\beta_2 - \beta_1 = 10\left(\log\left(\frac{I_2}{I_0} \div \frac{I_1}{I_0}\right)\right)$$

Apply the quotient law of logarithms.

$$\beta_2 - \beta_1 = 10\left(\log\left(\frac{I_2}{\cancel{I_0}} \times \frac{\cancel{I_0}}{I_1}\right)\right)$$

$$\beta_2 - \beta_1 = 10\left(\log \frac{I_2}{I_1}\right)$$

Decibel Scale



For each increase of 10 on the decibel scale, there is a tenfold increase in the intensity of sound.

Did You Know?

The unit used to measure the intensity of sound is the decibel (dB), named after Alexander Graham Bell, the inventor of the telephone. Bell was born in Scotland but lived most of his life in Canada.

Substitute $\beta_2 = 110$ and $\beta_1 = 85$ into the equation $\beta_2 - \beta_1 = 10 \log \frac{I_2}{I_1}$.

$$110 - 85 = 10 \log \frac{I_2}{I_1}$$

$$25 = 10 \log \frac{I_2}{I_1}$$

$$2.5 = \log \frac{I_2}{I_1}$$

$$10^{2.5} = \frac{I_2}{I_1}$$ Write in exponential form.

$$316 \approx \frac{I_2}{I_1}$$ What is another approach you could have used to find the ratio $\frac{I_2}{I_1}$?

The ratio of these two intensities is approximately 316. Hence, the maximum volume level of the portable media player is approximately 316 times as intense as the sound of a chainsaw.

b) The ratio of the intensity of sounds considered safe to the intensity of a whisper is 100 000 to 1. In the equation $\beta_2 - \beta_1 = 10 \log \frac{I_2}{I_1}$, substitute $\beta_1 = 20$ and $\frac{I_2}{I_1} = 100\,000$.

$$\beta_2 - 20 = 10 \log 100\,000$$
$$\beta_2 = 10 \log 100\,000 + 20$$
$$\beta_2 = 10 \log 10^5 + 20$$
$$\beta_2 = 10(5) + 20$$
$$\beta_2 = 70$$

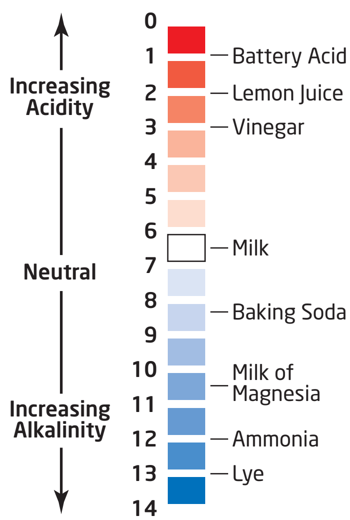
Sounds that are 70 dB or less pose no known risk of hearing loss, no matter how long they last.

Web Link

Some studies suggest that people exposed to excessive noise from leisure activities tend to develop hearing loss. The risk of noise-induced hearing loss depends on the sound level and the duration of the exposure. For more information, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

Your Turn

The pH scale is used to measure the acidity or alkalinity of a solution. The pH of a solution is defined as $pH = -\log [H^+]$, where $[H^+]$ is the hydrogen ion concentration in moles per litre (mol/L). A neutral solution, such as pure water, has a pH of 7. Solutions with a pH of less than 7 are acidic and solutions with a pH of greater than 7 are basic or alkaline. The closer the pH is to 0, the more acidic the solution is.



a) A common ingredient in cola drinks is phosphoric acid, the same ingredient found in many rust removers. A cola drink has a pH of 2.5. Milk has a pH of 6.6. How many times as acidic as milk is a cola drink?

b) An apple is 5 times as acidic as a pear. If a pear has a pH of 3.8, then what is the pH of an apple?

Key Ideas

- Let P be any real number, and M, N, and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Check Your Understanding

Practise

1. Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_7 xy^3\sqrt{z}$

b) $\log_5 (xyz)^8$

c) $\log \dfrac{x^2}{y\sqrt[3]{z}}$

d) $\log_3 x\sqrt{\dfrac{y}{z}}$

2. Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_{12} 24 - \log_{12} 6 + \log_{12} 36$

b) $3 \log_5 10 - \frac{1}{2} \log_5 64$

c) $\log_3 27\sqrt{3}$

d) $\log_2 72 - \frac{1}{2}(\log_2 3 + \log_2 27)$

3. Write each expression as a single logarithm in simplest form.

a) $\log_9 x - \log_9 y + 4 \log_9 z$

b) $\dfrac{\log_3 x}{2} - 2 \log_3 y$

c) $\log_6 x - \frac{1}{5}(\log_6 x + 2 \log_6 y)$

d) $\dfrac{\log x}{3} + \dfrac{\log y}{3}$

4. The original use of logarithms was to simplify calculations. Use the approximations shown on the right and the laws of logarithms to perform each calculation using only paper and pencil.

a) 1.44×1.2 $\quad\log 1.44 \approx 0.158\ 36$

b) $1.728 \div 1.2$ $\quad\log 1.2 \approx 0.079\ 18$

c) $\sqrt{1.44}$ $\quad\log 1.728 \approx 0.237\ 54$

5. Evaluate.

a) 3^k, where $k = \log_2 40 - \log_2 5$

b) 7^n, where $n = 3 \log_8 4$

Apply

6. To obtain the graph of $y = \log_2 8x$, you can either stretch or translate the graph of $y = \log_2 x$.

a) Describe the stretch you need to apply to the graph of $y = \log_2 x$ to result in the graph of $y = \log_2 8x$.

b) Describe the translation you need to apply to the graph of $y = \log_2 x$ to result in the graph of $y = \log_2 8x$.

7. Decide whether each equation is true or false. Justify your answer. Assume c, x, and y are positive real numbers and $c \neq 1$.

a) $\dfrac{\log_c x}{\log_c y} = \log_c x - \log_c y$

b) $\log_c (x + y) = \log_c x + \log_c y$

c) $\log_c c^n = n$

d) $(\log_c x)^n = n \log_c x$

e) $-\log_c \left(\dfrac{1}{x}\right) = \log_c x$

8. If $\log 3 = P$ and $\log 5 = Q$, write an algebraic expression in terms of P and Q for each of the following.

a) $\log \dfrac{3}{5}$

b) $\log 15$

c) $\log 3\sqrt{5}$

d) $\log \dfrac{25}{9}$

9. If $\log_2 7 = K$, write an algebraic expression in terms of K for each of the following.

a) $\log_2 7^6$

b) $\log_2 14$

c) $\log_2 (49 \times 4)$

d) $\log_2 \dfrac{\sqrt[5]{7}}{8}$

10. Write each expression as a single logarithm in simplest form. State any restrictions on the variable.

a) $\log_5 x + \log_5 \sqrt{x^3} - 2 \log_5 x$

b) $\log_{11} \dfrac{x}{\sqrt{x}} + \log_{11} \sqrt{x^5} - \dfrac{7}{3} \log_{11} x$

11. Write each expression as a single logarithm in simplest form. State any restrictions on the variable.

a) $\log_2 (x^2 - 25) - \log_2 (3x - 15)$

b) $\log_7 (x^2 - 16) - \log_7 (x^2 - 2x - 8)$

c) $2 \log_8 (x + 3) - \log_8 (x^2 + x - 6)$

12. Show that each equation is true for $c > 0$ and $c \neq 1$.

a) $\log_c 48 - (\log_c 3 + \log_c 2) = \log_c 8$

b) $7 \log_c 4 = 14 \log_c 2$

c) $\dfrac{1}{2}(\log_c 2 + \log_c 6) = \log_c 2 + \log_c \sqrt{3}$

d) $\log_c (5c)^2 = 2(\log_c 5 + 1)$

13. Sound intensity, β, in decibels is defined as $\beta = 10 \log \left(\dfrac{I}{I_0}\right)$, where I is the intensity of the sound measured in watts per square metre (W/m^2) and I_0 is 10^{-12} W/m^2, the threshold of hearing.

a) The sound intensity of a hairdryer is 0.000 01 W/m^2. Find its decibel level.

b) A fire truck siren has a decibel level of 118 dB. City traffic has a decibel level of 85 dB. How many times as loud as city traffic is the fire truck siren?

c) The sound of Elly's farm tractor is 63 times as intense as the sound of her car. If the decibel level of the car is 80 dB, what is the decibel level of the farm tractor?

14. Abdi incorrectly states, "A noise of 20 dB is twice as loud as a noise of 10 dB." Explain the error in Abdi's reasoning.

15. The term *decibel* is also used in electronics for current and voltage ratios. Gain is defined as the ratio between the signal coming in and the signal going out. The gain, G, in decibels, of an amplifier is defined as $G = 20 \log \dfrac{V}{V_i}$, where V is the voltage output and V_i is the voltage input. If the gain of an amplifier is 24 dB when the voltage input is 0.2 V, find the voltage output, V. Answer to the nearest tenth of a volt.

16. The logarithmic scale used to express the pH of a solution is pH = $-\log [H^+]$, where $[H^+]$ is the hydrogen ion concentration, in moles per litre (mol/L).

   a) Lactic acidosis is medical condition characterized by elevated lactates and a blood pH of less than 7.35. A patient is severely ill when his or her blood pH is 7.0. Find the hydrogen ion concentration in a patient with a blood pH of 7.0.

   b) Acid rain is caused when compounds from combustion react with water in the atmosphere to produce acids. It is generally accepted that rain is acidic if its pH is less than 5.3. The average pH of rain in some regions of Ontario is about 4.5. How many times as acidic as normal rain with a pH of 5.6 is acid rain with a pH of 4.5?

   c) The hair conditioner that Alana uses is 500 times as acidic as the shampoo she uses. If the shampoo has a pH of 6.1, find the pH of the conditioner.

17. The change in velocity, Δv, in kilometres per second, of a rocket with an exhaust velocity of 3.1 km/s can be found using the Tsiolkovsky rocket equation $\Delta v = \frac{3.1}{0.434}(\log m_0 - \log m_f)$, where m_0 is the initial total mass and m_f is the final total mass, in kilograms, after a fuel burn. Find the change in the velocity of the rocket if the mass ratio, $\frac{m_0}{m_f}$, is 1.06. Answer to the nearest hundredth of a kilometre per second.

Extend

18. Graph the functions $y = \log x^2$ and $y = 2 \log x$ on the same coordinate grid.

   a) How are the graphs alike? How are they different?

   b) Explain why the graphs are not identical.

   c) Although the functions $y = \log x^2$ and $y = 2 \log x$ are not the same, the equation $\log x^2 = 2 \log x$ is true. This is because the variable x in the equation is restricted to values for which both logarithms are defined. What is the restriction on x in the equation?

19. **a)** Prove the change of base formula, $\log_c x = \frac{\log_d x}{\log_d c}$, where c and d are positive real numbers other than 1.

   b) Apply the change of base formula for base $d = 10$ to find the approximate value of $\log_2 9.5$ using common logarithms. Answer to four decimal places.

   c) The Krumbein phi (φ) scale is used in geology to classify the particle size of natural sediments such as sand and gravel. The formula for the φ-value may be expressed as $\varphi = -\log_2 D$, where D is the diameter of the particle, in millimetres. The φ-value can also be defined using a common logarithm. Express the formula for the φ-value as a common logarithm.

   d) How many times the diameter of medium sand with a φ-value of 2 is the diameter of a pebble with a φ-value of -5.7? Determine the answer using both versions of the φ-value formula from part c).

20. Prove each identity.

   a) $\log_{q^3} p^3 = \log_q p$

   b) $\frac{1}{\log_p 2} - \frac{1}{\log_q 2} = \log_2 \frac{p}{q}$

   c) $\frac{1}{\log_q p} + \frac{1}{\log_q p} = \frac{1}{\log_{q^2} p}$

   d) $\log_{\frac{1}{q}} p = \log_q \frac{1}{p}$

Create Connections

C1 Describe how you could obtain the graph of each function from the graph of $y = \log x$.

a) $y = \log x^3$

b) $y = \log (x + 2)^5$

c) $y = \log \dfrac{1}{x}$

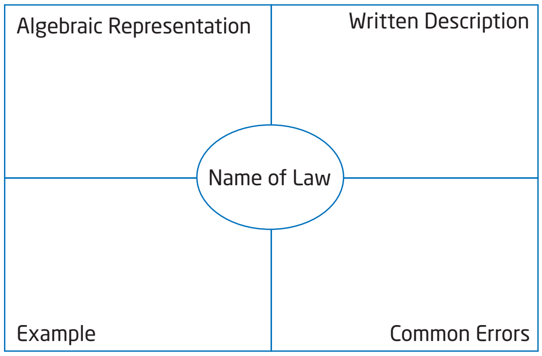
d) $y = \log \dfrac{1}{\sqrt{x - 6}}$

C2 Evaluate $\log_2 \left(\sin \dfrac{\pi}{4}\right) + \log_2 \left(\sin \dfrac{3\pi}{4}\right)$.

C3 **a)** What is the common difference, d, in the arithmetic series $\log 2 + \log 4 + \log 8 + \log 16 + \log 32$?

b) Express the sum of the series as a multiple of the common difference.

C4 Copy the Frayer Model template shown for each law of logarithms. In the appropriate space, give the name of the law, an algebraic representation, a written description, an example, and common errors.



Project Corner

Modelling Data

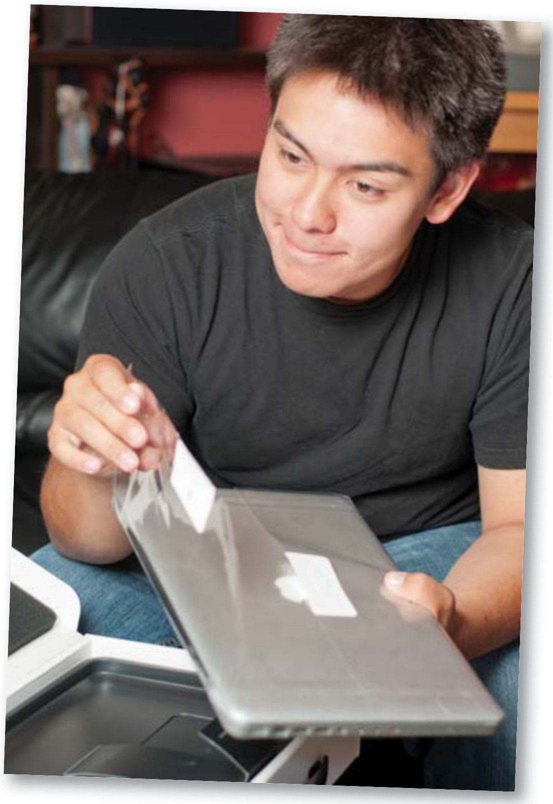
The table shows box office receipts for a popular new movie.

- Determine the equation of a logarithmic function of the form $y = 20 \log_{1.3} (x - h) + k$ that fits the data.
- Determine the equation of an exponential function of the form $y = -104(0.74)^{x-h} + k$ that fits the data.
- Compare the logarithmic function to the exponential function. Is one model better than the other? Explain.

Week	Cumulative Box Office Revenue (millions of dollars)
1	70
2	144
3	191
4	229
5	256
6	275
7	291
8	304
9	313
10	320
11	325
12	328
13	330
14	332
15	334
16	335
17	335
18	336
19	337

8.4

Logarithmic and Exponential Equations



Focus on...

- solving a logarithmic equation and verifying the solution
- explaining why a value obtained in solving a logarithmic equation may be extraneous
- solving an exponential equation in which the bases are not powers of one another
- solving a problem that involves exponential growth or decay
- solving a problem that involves the application of exponential equations to loans, mortgages, and investments
- solving a problem by modelling a situation with an exponential or logarithmic equation

Change is taking place in our world at a pace that is unprecedented in human history. Think of situations in your life that show exponential growth.

Imagine you purchase a computer with 1 TB (terabyte) of available disk space. One terabyte equals 1 048 576 MB (megabytes). On the first day you store 1 MB (megabyte) of data on the disk space. On each successive day you store twice the data stored on the previous day. Predict on what day you will run out of disk space.

The table below shows how the disk space fills up.

On what day will you realise that you are running out of disk space?

Notice that the amount stored on any given day, after the first day, exceeds the total amount stored on all the previous days.

Computer Disk Space					
Day	Data Stored (MB)	Space Used (MB)	Space Unused (MB)	Percent Used	Percent Unused
1	1	1	1048575	0.0	100.0
2	2	3	1048573	0.0	100.0
3	4	7	1048569	0.0	100.0
4	8	15	1048561	0.0	100.0
5	16	31	1048545	0.0	100.0
6	32	63	1048513	0.0	100.0
7	64	127	1048449	0.0	100.0
8	128	255	1048321	0.0	100.0
9	256	511	1048065	0.0	100.0
10	512	1023	1047553	0.1	99.9
11	1024	2047	1046529	0.2	99.8
12	2048	4095	1044481	0.4	99.6
13	4096	8191	1040385	0.8	99.2
14	8192	16383	1032193	1.6	98.4
15	16384	32767	1015809	3.1	96.9
16	32768	65535	983041	6.2	93.8
17	65536	131071	917505	12.5	87.5
18	131072	262143	786433	25.0	75.0
19	262144	524287	524289	50.0	50.0
20	524288	1048575	1	100.0	0.0

Suppose you purchase an external hard drive with an additional 15 TB of disk storage space. For how long can you continue doubling the amount stored?

You can find the amount of data stored on a certain day by using logarithms to solve an exponential equation. Solving problems involving exponential and logarithmic equations helps us to understand and shape our ever-changing world.

Investigate Logarithmic and Exponential Equations

Part A: Explore Logarithmic Equations

Consider the **logarithmic equation** $2 \log x = \log 36$.

logarithmic equation

- an equation containing the logarithm of a variable

1. Use the following steps to solve the equation.

a) Apply one of the laws of logarithms to the left side of the equation.

b) Describe how you might solve the resulting equation.

c) Determine two values of x that satisfy the rewritten equation in part a).

2. a) Describe how you could solve the original equation graphically.

b) Use your description to solve the original equation graphically.

Reflect and Respond

3. What value or values of x satisfy the original equation? Explain.

Part B: Explore Exponential Equations

Adam and Sarah are asked to solve the exponential equation $2(25^{x+1}) = 250$. Each person uses a different method.

	Adam's Method	Sarah's Method
	$2(25^{x+1}) = 250$	$2(25^{x+1}) = 250$
Step 1	$25^{x+1} = 125$	$25^{x+1} = 125$
Step 2	$(5^2)^{x+1} = 5^3$	$\log 25^{x+1} = \log 125$
Step 3	$5^{2(x+1)} = 5^3$	$(x + 1) \log 25 = \log 125$
Step 4	$2(x + 1) = 3$ $2x + 2 = 3$ $2x = 1$ $x = 0.5$	$x \log 25 + \log 25 = \log 125$ $x \log 25 = \log 125 - \log 25$ $x = \dfrac{\log\left(\frac{125}{25}\right)}{\log 25}$ $= \dfrac{\log 5}{\log 5^2}$ $= \dfrac{\cancelto{1}{\log 5}}{2 \cancelto{1}{\log 5}}$ $= \dfrac{1}{2}$

4. Explain each step in each student's work.

5. What is another way that Adam could have completed step 4 of his work? Show another way Sarah could have completed step 4 of her work.

Reflect and Respond

6. Which person's method do you prefer? Explain why.

7. What types of exponential equations could be solved using Adam's method? What types of exponential equations could not be solved using Adam's method and must be solved using Sarah's method? Explain.

8. Sarah used common logarithms in step 2 of her work. Could she instead have used logarithms to another base? Justify your answer.

Link the Ideas

The following equality statements are useful when solving an exponential equation or a logarithmic equation.

Given $c, L, R > 0$ and $c \neq 1$,
- if $\log_c L = \log_c R$, then $L = R$
- if $L = R$, then $\log_c L = \log_c R$

Proof

Let $\log_c L = \log_c R$.

$c^{\log_c R} = L$ Write in exponential form.

$R = L$ Apply the inverse property of logarithms, $c^{\log_c x} = x$, where $x > 0$.

When solving a logarithmic equation, identify whether any roots are extraneous by substituting into the original equation and determining whether all the logarithms are defined. The logarithm of zero or a negative number is undefined.

Example 1

Solve Logarithmic Equations

Solve.

a) $\log_6 (2x - 1) = \log_6 11$ **b)** $\log (8x + 4) = 1 + \log (x + 1)$

c) $\log_2 (x + 3)^2 = 4$

Solution

a) **Method 1: Solve Algebraically**

The following statement is true for $c, L, R > 0$ and $c \neq 1$.

If $\log_c L = \log_c R$, then $L = R$.

Hence,

$$\begin{aligned} \log_6 (2x - 1) &= \log_6 11 \\ 2x - 1 &= 11 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

The equation $\log_6 (2x - 1) = \log_6 11$ is defined when $2x - 1 > 0$. This occurs when $x > \frac{1}{2}$. Since the value of x satisfies this restriction, the solution is $x = 6$.

Check $x = 6$ in the original equation, $\log_6 (2x - 1) = \log_6 11$.

Left Side	Right Side
$\log_6 (2x - 1)$	$\log_6 11$
$= \log_6 (2(6) - 1)$	
$= \log_6 11$	

Left Side = Right Side

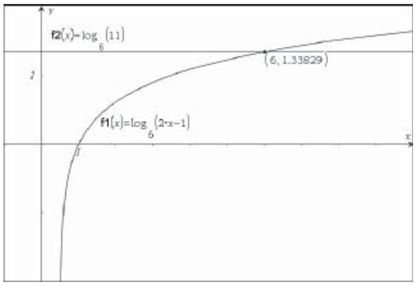
Method 2: Solve Graphically

Find the graphical solution to the system of equations:

$y = \log_6 (2x - 1)$

$y = \log_6 11$

The x-coordinate at the point of intersection of the graphs of the functions is the solution, $x = 6$.



How could you have used the graph of $y = \log_6 (2x - 1) - \log_6 11$ to solve the equation?

b)

$$\log (8x + 4) = 1 + \log (x + 1)$$

$$\log_{10} (8x + 4) = 1 + \log_{10} (x + 1)$$

$$\log_{10} (8x + 4) - \log_{10} (x + 1) = 1$$

Isolate the logarithmic terms on one side of the equation.

$$\log_{10} \frac{8x + 4}{x + 1} = 1$$

Apply the quotient law for logarithms.

Select a strategy to solve for x.

Method 1: Express Both Sides of the Equation as Logarithms

$$\log_{10} \frac{8x + 4}{x + 1} = 1$$

$$\log_{10} \frac{8x + 4}{x + 1} = \log_{10} 10^1$$

Use the property $\log_b b^n = n$ to substitute $\log_{10} 10^1$ for 1.

$$\frac{8x + 4}{x + 1} = 10$$

Use the property that if $\log_c L = \log_c R$, then $L = R$.

$$8x + 4 = 10(x + 1)$$

Multiply both sides of the equation by $x + 1$, the lowest common denominator (LCD).

$$8x + 4 = 10x + 10$$

$$-6 = 2x$$

Solve the linear equation.

$$-3 = x$$

Method 2: Convert to Exponential Form

$$\log_{10} \frac{8x + 4}{x + 1} = 1$$

$$\frac{8x + 4}{x + 1} = 10^1$$

Write in exponential form.

$$8x + 4 = 10(x + 1)$$

Multiply both sides of the equation by the LCD, $x + 1$.

$$8x + 4 = 10x + 10$$

$$-6 = 2x$$

Solve the linear equation.

$$-3 = x$$

The solution $x = -3$ is extraneous. When -3 is substituted for x in the original equation, both $\log (8x + 4)$ and $\log (x + 1)$ are undefined. Hence, there is no solution to the equation.

Why is the logarithm of a negative number undefined?

c) $\log_2 (x + 3)^2 = 4$

$(x + 3)^2 = 2^4$

If the power law of logarithms was used, how would this affect the solution?

$x^2 + 6x + 9 = 16$

$x^2 + 6x - 7 = 0$

$(x + 7)(x - 1) = 0$

$x = -7$ or $x = 1$

When either -7 or 1 is substituted for x in the original equation, $\log_2 (x + 3)^2$ is defined.

Check:

Substitute $x = -7$ and $x = 1$ in the original equation, $\log_2 (x + 3)^2 = 4$.

When $x = -7$:

Left Side	Right Side
$\log_2 (x + 3)^2$	4
$= \log_2 (-7 + 3)^2$	
$= \log_2 (-4)^2$	
$= \log_2 16$	
$= 4$	

Left Side = Right Side

When $x = 1$:

Left Side	Right Side
$\log_2 (x + 3)^2$	4
$= \log_2 (1 + 3)^2$	
$= \log_2 (4)^2$	
$= \log_2 16$	
$= 4$	

Left Side = Right Side

Your Turn

Solve.

a) $\log_7 x + \log_7 4 = \log_7 12$

b) $\log_2 (x - 6) = 3 - \log_2 (x - 4)$

c) $\log_3 (x^2 - 8x)^5 = 10$

Example 2

Solve Exponential Equations Using Logarithms

Solve. Round your answers to two decimal places.

a) $4^x = 605$

b) $8(3^{2x}) = 568$

c) $4^{2x - 1} = 3^{x + 2}$

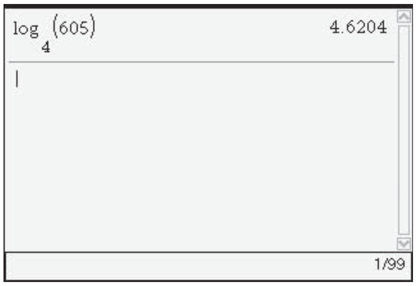
Solution

a) **Method 1: Take Common Logarithms of Both Sides**

$$4^x = 605$$
$$\log 4^x = \log 605$$
$$x \log 4 = \log 605$$
$$x = \frac{\log 605}{\log 4}$$
$$x \approx 4.62$$

Method 2: Convert to Logarithmic Form

$$4^x = 605$$
$$\log_4 605 = x$$
$$4.62 \approx x$$



Which method do you prefer? Explain why.

Check $x \approx 4.62$ in the original equation, $4^x = 605$.

Left Side	Right Side
$4^{4.62}$	605
≈ 605	

Left Side $\approx$ Right Side

b)
$$8(3^{2x}) = 568$$
$$3^{2x} = 71$$
$$\log 3^{2x} = \log 71$$
$$2x(\log 3) = \log 71$$
$$x = \frac{\log 71}{2 \log 3}$$
$$x \approx 1.94$$

Explain why $8(3^{2x})$ cannot be expressed as 24^{2x}.

Explain the steps in this solution. How else could you have used logarithms to solve for x?

Check $x \approx 1.94$ in the original equation, $8(3^{2x}) = 568$.

Left Side	Right Side
$8(3^{2(1.94)})$	568
≈ 568	

Left Side $\approx$ Right Side

c)
$$4^{2x-1} = 3^{x+2}$$
$$\log 4^{2x-1} = \log 3^{x+2}$$
$$(2x - 1)\log 4 = (x + 2)\log 3$$
$$2x \log 4 - \log 4 = x \log 3 + 2 \log 3$$
$$2x \log 4 - x \log 3 = 2 \log 3 + \log 4$$
$$x(2 \log 4 - \log 3) = 2 \log 3 + \log 4$$
$$x = \frac{2 \log 3 + \log 4}{2 \log 4 - \log 3}$$
$$x \approx 2.14$$

Check $x \approx 2.14$ in the original equation, $4^{2x-1} = 3^{x+2}$.

Left Side	Right Side
$4^{2(2.14)-1}$	$3^{2.14+2}$
≈ 94	≈ 94

Left Side $\approx$ Right Side

What is another way you can verify the solution?

Your Turn

Solve. Round answers to two decimal places.

a) $2^x = 2500$ **b)** $5^{x-3} = 1700$ **c)** $6^{3x+1} = 8^{x+3}$

Example 3

Model a Situation Using a Logarithmic Equation

Palaeontologists can estimate the size of a dinosaur from incomplete skeletal remains. For a carnivorous dinosaur, the relationship between the length, s, in metres, of the skull and the body mass, m, in kilograms, can be expressed using the logarithmic equation $3.6022 \log s = \log m - 3.4444$. Determine the body mass, to the nearest kilogram, of an Albertosaurus with a skull length of 0.78 m.

Albertosaurus display, Royal Tyrrell Museum, Drumheller, Alberta

Did You Know?

Albertosaurus was the top predator in the semi-tropical Cretaceous ecosystem, more than 70 million years ago. It was smaller than its close relative Tyrannosaurus rex, which lived a few million years later. The first Albertosaurus was discovered by Joseph B. Tyrrell, a geologist searching for coal deposits in the Red Deer River valley, in 1884. Since then, more than 30 Albertosauruses have been discovered in western North America.

Solution

Substitute $s = 0.78$ into the equation $3.6022 \log s = \log m - 3.4444$.

$$\begin{aligned} 3.6022 \log_{10} 0.78 &= \log_{10} m - 3.4444 \\ 3.6022 \log_{10} 0.78 + 3.4444 &= \log_{10} m \\ 3.0557 &\approx \log_{10} m \\ 10^{3.0557} &\approx m \\ 1137 &\approx m \end{aligned}$$

The mass of the Albertosaurus was approximately 1137 kg.

Your Turn

To the nearest hundredth of a metre, what was the skull length of a Tyrannosaurus rex with an estimated body mass of 5500 kg?

Example 4

Solve a Problem Involving Exponential Growth and Decay

When an animal dies, the amount of radioactive carbon-14 (C-14) in its bones decreases. Archaeologists use this fact to determine the age of a fossil based on the amount of C-14 remaining.
The half-life of C-14 is 5730 years.


Buffalo skull display, Head-Smashed-In buffalo Jump Visitor Centre, near Fort McLeod, Alberta

Head-Smashed-In Buffalo Jump in southwestern Alberta is recognized as the best example of a buffalo jump in North America. The oldest bones unearthed at the site had 49.5% of the C-14 left. How old were the bones when they were found?

Did You Know?

First Nations hunters used a variety of strategies to harvest the largest land mammal in North America, the buffalo. The most effective method for securing large quantities of food was the buffalo jump. The extraordinary amount of work required to plan, coordinate, and implement a successful harvest demonstrates First Nations Peoples' ingenuity, communal cooperation, and organizational skills that enabled them to utilize this primary resource in a sustainable manner for millennia.

Solution

Carbon-14 decays by one half for each 5730-year interval. The mass, m, remaining at time t can be found using the relationship $m(t) = m_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$, where m_0 is the original mass.

Since 49.5% of the C-14 remains after t years, substitute $0.495m_0$ for $m(t)$ in the formula $m(t) = m_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$.

$$\begin{aligned}
0.495m_0 &= m_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}\\
0.495 &= 0.5^{\frac{t}{5730}}\\
\log 0.495 &= \log 0.5^{\frac{t}{5730}}\\
\log 0.495 &= \frac{t}{5730}\log 0.5\\
\frac{5730 \log 0.495}{\log 0.5} &= t\\
5813 &\approx t
\end{aligned}$$

Instead of taking the common logarithm of both sides, you could have converted from exponential form to logarithmic form. Try this. Which approach do you prefer? Why?

The oldest buffalo bones found at Head-Smashed-In Buffalo Jump date to about 5813 years ago. The site has been used for at least 6000 years.

Your Turn

The rate at which an organism duplicates is called its doubling period. The general equation is $N(t) = N_0(2)^{\frac{t}{d}}$, where N is the number present after time t, N_0 is the original number, and d is the doubling period. *E. coli* is a rod-shaped bacterium commonly found in the intestinal tract of warm-blooded animals. Some strains of *E. coli* can cause serious food poisoning in humans. Suppose a biologist originally estimates the number of *E. coli* bacteria in a culture to be 1000. After 90 min, the estimated count is 19 500 bacteria. What is the doubling period of the *E. coli* bacteria, to the nearest minute?

Key Ideas

- When solving a logarithmic equation algebraically, start by applying the laws of logarithms to express one side or both sides of the equation as a single logarithm.
- Some useful properties are listed below, where c, L, $R > 0$ and $c \neq 1$.
  - If $\log_c L = \log_c R$, then $L = R$.
  - The equation $\log_c L = R$ can be written with logarithms on both sides of the equation as $\log_c L = \log_c c^R$.
  - The equation $\log_c L = R$ can be written in exponential form as $L = c^R$.
  - The logarithm of zero or a negative number is undefined. To identify whether a root is extraneous, substitute the root into the original equation and check whether all of the logarithms are defined.
- You can solve an exponential equation algebraically by taking logarithms of both sides of the equation. If $L = R$, then $\log_c L = \log_c R$, where c, L, $R > 0$ and $c \neq 1$. Then, apply the power law for logarithms to solve for an unknown.
- You can solve an exponential equation or a logarithmic equation using graphical methods.
- Many real-world situations can be modelled with an exponential or a logarithmic equation. A general model for many problems involving exponential growth or decay is

$$\text{final quantity} = \text{initial quantity} \times (\text{change factor})^{\text{number of changes}}$$

Check Your Understanding

Practise

1. Solve. Give exact answers.

a) $15 = 12 + \log x$

b) $\log_5 (2x - 3) = 2$

c) $4 \log_3 x = \log_3 81$

d) $2 = \log (x - 8)$

2. Solve for x. Give your answers to two decimal places.

a) $4(7^x) = 92$

b) $2^{\frac{x}{3}} = 11$

c) $6^{x-1} = 271$

d) $4^{2x+1} = 54$

3. Hamdi algebraically solved the equation $\log_3 (x - 8) - \log_3 (x - 6) = 1$ and found $x = 5$ as a possible solution. The following shows Hamdi's check for $x = 5$.

Left Side	Right Side
$\log_3 \frac{x-8}{x-6}$	1
$= \log_3 \frac{5-8}{5-6}$	
$= \log_3 3$	
$= 1$	

Left Side = Right Side

Do you agree with Hamdi's check? Explain why or why not.

4. Determine whether the possible roots listed are extraneous to the logarithmic equation given.

a) $\log_7 x + \log_7 (x - 1) = \log_7 4x$
possible roots: $x = 0$, $x = 5$

b) $\log_6 (x^2 - 24) - \log_6 x = \log_6 5$
possible roots: $x = 3$, $x = -8$

c) $\log_3 (x + 3) + \log_3 (x + 5) = 1$
possible roots: $x = -2$, $x = -6$

d) $\log_2 (x - 2) = 2 - \log_2 (x - 5)$
possible roots: $x = 1$, $x = 6$

5. Solve for x.

a) $2 \log_3 x = \log_3 32 + \log_3 2$

b) $\frac{3}{2} \log_7 x = \log_7 125$

c) $\log_2 x - \log_2 3 = 5$

d) $\log_6 x = 2 - \log_6 4$

Apply

6. Three students each attempted to solve a different logarithmic equation. Identify and describe any error in each person's work, and then correctly solve the equation.

a) *Rubina's work:*

$$\log_6 (2x + 1) - \log_6 (x - 1) = \log_6 5$$
$$\log_6 (x + 2) = \log_6 5$$
$$x + 2 = 5$$
$$x = 3$$

The solution is $x = 3$.

b) *Ahmed's work:*

$$2 \log_5 (x + 3) = \log_5 9$$
$$\log_5 (x + 3)^2 = \log_5 9$$
$$(x + 3)^2 = 9$$
$$x^2 + 6x + 9 = 9$$
$$x(x + 6) = 0$$
$$x = 0 \text{ or } x = -6$$

There is no solution.

c) *Jennifer's work:*

$$\log_2 x + \log_2 (x + 2) = 3$$
$$\log_2 (x(x + 2)) = 3$$
$$x(x + 2) = 3$$
$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0$$
$$x = -3 \text{ or } x = 1$$

The solution is $x = 1$.

7. Determine the value of x. Round your answers to two decimal places.

a) $7^{2x} = 2^{x+3}$

b) $1.6^{x-4} = 5^{3x}$

c) $9^{2x-1} = 71^{x+2}$

d) $4(7^{x+2}) = 9^{2x-3}$

8. Solve for x.

a) $\log_5 (x - 18) - \log_5 x = \log_5 7$

b) $\log_2 (x - 6) + \log_2 (x - 8) = 3$

c) $2 \log_4 (x + 4) - \log_4 (x + 12) = 1$

d) $\log_3 (2x - 1) = 2 - \log_3 (x + 1)$

e) $\log_2 \sqrt{x^2 + 4x} = \frac{5}{2}$

9. The apparent magnitude of a celestial object is how bright it appears from Earth. The absolute magnitude is its brightness as it would seem from a reference distance of 10 parsecs (pc). The difference between the apparent magnitude, m, and the absolute magnitude, M, of a celestial object can be found using the equation $m - M = 5 \log d - 5$, where d is the distance to the celestial object, in parsecs. Sirius, the brightest star visible at night, has an apparent magnitude of -1.44 and an absolute magnitude of 1.45.

a) How far is Sirius from Earth in parsecs?

b) Given that 1 pc is approximately 3.26 light years, what is the distance in part a) in light years?

10. Small animal characters in animated features are often portrayed with big endearing eyes. In reality, the eye size of many vertebrates is related to body mass by the logarithmic equation $\log E = \log 10.61 + 0.1964 \log m$, where E is the eye axial length, in millimetres, and m is the body mass, in kilograms. To the nearest kilogram, predict the mass of a mountain goat with an eye axial length of 24 mm.

11. A remote lake that previously contained no northern pike is stocked with these fish. The population, P, of northern pike after t years can be determined by the equation $P = 10\ 000(1.035)^t$.

a) How many northern pike were put into the lake when it was stocked?

b) What is the annual growth rate, as a percent?

c) How long will it take for the number of northern pike in the lake to double?

12. The German astronomer Johannes Kepler developed three major laws of planetary motion. His third law can be expressed by the equation $\log T = \frac{3}{2} \log d - 3.263$, where T is the time, in Earth years, for the planet to revolve around the sun and d is the average distance, in millions of kilometres, from the sun.

a) Pluto is on average 5906 million kilometres from the sun. To the nearest Earth year, how long does it take Pluto to revolve around the sun?

b) Mars revolves around the sun in 1.88 Earth years. How far is Mars from the sun, to the nearest million kilometres?

13. The compound interest formula is $A = P(1 + i)^n$, where A is the future amount, P is the present amount or principal, i is the interest rate per compounding period expressed as a decimal, and n is the number of compounding periods. All interest rates are annual percentage rates (APR).

a) David inherits \$10 000 and invests in a guaranteed investment certificate (GIC) that earns 6%, compounded semi-annually. How long will it take for the GIC to be worth \$11 000?

b) Linda used a credit card to purchase a \$1200 laptop computer. The rate of interest charged on the overdue balance is 28% per year, compounded daily. How many days is Linda's payment overdue if the amount shown on her credit card statement is \$1241.18?

c) How long will it take for money invested at 5.5%, compounded semi-annually, to triple in value?

14. A mortgage is a long-term loan secured by property. A mortgage with a present value of \$250 000 at a 7.4% annual percentage rate requires semi-annual payments of \$10 429.01 at the end of every 6 months. The formula for the present value, PV, of the mortgage is $PV = \frac{R[1 - (1 + i)^{-n}]}{i}$, where n is the number of equal periodic payments of R dollars and i is the interest rate per compounding period, as a decimal. After how many years will the mortgage be completely paid off?

15. Swedish researchers report that they have discovered the world's oldest living tree. The spruce tree's roots were radiocarbon dated and found to have 31.5% of their carbon-14 (C-14) left. The half-life of C-14 is 5730 years. How old was the tree when it was discovered?

Norway spruce, Dalarna, Sweden

16. Radioisotopes are used to diagnose various illnesses. Iodine-131 (I-131) is administered to a patient to diagnose thyroid gland activity. The original dosage contains 280 MBq of I-131. If none is lost from the body, then after 6 h there are 274 MBq of I-131 in the patient's thyroid. What is the half-life of I-131, to the nearest day?

Did You Know?

The SI unit used to measure radioactivity is the becquerel (Bq), which is one particle emitted per second from a radioactive source. Commonly used multiples are kilobecquerel (kBq), for 10^3 Bq, and megabecquerel (MBq), for 10^6 Bq.

17. The largest lake lying entirely within Canada is Great Bear Lake, in the Northwest Territories. On a summer day, divers find that the light intensity is reduced by 4% for every metre below the water surface. To the nearest tenth of a metre, at what depth is the light intensity 25% of the intensity at the surface?

18. If $\log_3 81 = x - y$ and $\log_2 32 = x + y$, determine the values of x and y.

Extend

19. Find the error in each.

a) $\log 0.1 < 3 \log 0.1$

Since $3 \log 0.1 = \log 0.1^3$,
$\log 0.1 < \log 0.1^3$
$\log 0.1 < \log 0.001$
Therefore, $0.1 < 0.001$.

b) $\frac{1}{5} > \frac{1}{25}$

$\log \frac{1}{5} > \log \frac{1}{25}$

$\log \frac{1}{5} > \log \left(\frac{1}{5}\right)^2$

$\log \frac{1}{5} > 2 \log \frac{1}{5}$

Therefore, $1 > 2$.

20. Solve for x.

a) $x^{\frac{2}{\log x}} = x$

b) $\log x^{\log x} = 4$

c) $(\log x)^2 = \log x^2$

21. Solve for x.

a) $\log_4 x + \log_2 x = 6$

b) $\log_3 x - \log_{27} x = \frac{4}{3}$

22. Determine the values of x that satisfy the equation $(x^2 + 3x - 9)^{2x-8} = 1$.

Create Connections

C1 Fatima started to solve the equation $8(2^x) = 512$, as shown.

$$8(2^x) = 512$$
$$\log 8(2^x) = \log 512$$
$$\log 8 + \log 2^x = \log 512$$

a) Copy and complete the solution using Fatima's approach.

b) Suggest another approach Fatima could have used to solve the equation. Compare the different approaches among classmates.

c) Which approach do you prefer? Explain why.

C2 The general term, t_n, of a geometric sequence is $t_n = t_1 r^{n-1}$, where t_1 is the first term of the sequence, n is the number of terms, and r is the common ratio. Determine the number of terms in the geometric sequence 4, 12, 36, …, 708 588.

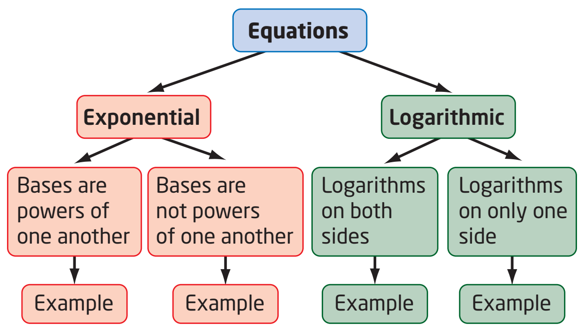
C3 The sum, S_n, of the first n terms of a geometric series can be found using the formula $S_n = \frac{t_1(r^n - 1)}{r - 1}$, $r \neq 1$, where t_1 is the first term and r is the common ratio. The sum of the first n terms in the geometric series $8192 + 4096 + 2048 + \cdots$ is 16 383. Determine the value of n.

C4 Solve for x, $0 \leq x \leq 2\pi$.

a) $2 \log_2 (\cos x) + 1 = 0$

b) $\log (\sin x) + \log (2 \sin x - 1) = 0$

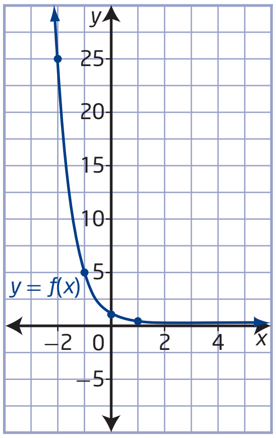
C5 Copy the concept chart. Provide worked examples in the last row.



Chapter 8 Review

8.1 Understanding Logarithms, pages 372–382

1. A graph of $f(x) = 0.2^x$ is shown.



a) Make a copy of the graph and on the same grid sketch the graph of $y = f^{-1}(x)$.

b) Determine the following characteristics of $y = f^{-1}(x)$.

i) the domain and range

ii) the x-intercept, if it exists

iii) the y-intercept, if it exists

iv) the equation of the asymptote

c) State the equation of $f^{-1}(x)$.

2. The point (2, 16) is on the graph of the inverse of $y = \log_c x$. What is the value of c?

3. Explain why the value of $\log_2 24$ must be between 4 and 5.

4. Determine the value of x.

a) $\log_{125} x = \frac{2}{3}$

b) $\log_9 \frac{1}{81} = x$

c) $\log_3 27\sqrt{3} = x$

d) $\log_x 8 = \frac{3}{4}$

e) $6^{\log x} = \frac{1}{36}$

5. The formula for the Richter magnitude, M, of an earthquake is $M = \log \frac{A}{A_0}$, where A is the amplitude of the ground motion and A_0 is the amplitude of a standard earthquake. In 2011, an earthquake with a Richter magnitude of 9.0 struck off the east coast of Japan. In the aftermath of the earthquake, a 10-m-tall tsunami swept across the country. Hundreds of aftershocks came in the days that followed, some with magnitudes as great as 7.4 on the Richter scale. How many times as great as the seismic shaking of the large aftershock was the shaking of the initial earthquake?

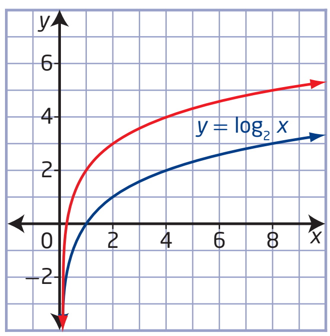
8.2 Transformations of Logarithmic Functions, pages 383–391

6. The graph of $y = \log_4 x$ is
- stretched horizontally about the y-axis by a factor of $\frac{1}{2}$
- reflected in the x-axis
- translated 5 units down

a) Sketch the graph of the transformed image.

b) If the equation of the transformed image is written in the form $y = a \log_c (b(x - h)) + k$, determine the values of a, b, c, h, and k.

7. The red graph is a stretch of the blue graph. Determine the equation of the red graph.



8. Describe, in order, a series of transformations that could be applied to the graph of $y = \log_5 x$ to draw the graph of each function.

a) $y = -\log_5 (3(x - 12)) + 2$

b) $y + 7 = \frac{\log_5 (6 - x)}{4}$

9. Identity the following characteristics of the graph of the function $y = 3 \log_2 (x + 8) + 6$.

a) the equation of the asymptote

b) the domain and range

c) the y-intercept

d) the x-intercept

10. Starting at the music note A, with a frequency of 440 Hz, the frequency of the other musical notes can be determined using the function $n = 12 \log_2 \frac{f}{440}$, where n is the number of notes away from A.

a) Describe how the function is transformed from $n = \log_2 f$.

b) How many notes above A is the note D, if D has a frequency of 587.36 Hz?

c) Find the frequency of F, located eight notes above A. Answer to the nearest hundredth of a hertz.

8.3 Laws of Logarithms, pages 392–403

11. Write each expression in terms of the individual logarithms of x, y, and z.

a) $\log_5 \frac{x^5}{y\sqrt[3]{z}}$

b) $\log \sqrt{\frac{xy^2}{z}}$

12. Write each expression as a single logarithm in simplest form.

a) $\log x - 3 \log y + \frac{2}{3} \log z$

b) $\log x - \frac{1}{2}(\log y + 3 \log z)$

13. Write each expression as a single logarithm in simplest form. State any restrictions.

a) $2 \log x + 3 \log \sqrt{x} - \log x^3$

b) $\log (x^2 - 25) - 2 \log (x + 5)$

14. Use the laws of logarithms to simplify and then evaluate each expression.

a) $\log_6 18 - \log_6 2 + \log_6 4$

b) $\log_4 \sqrt{12} + \log_4 \sqrt{9} - \log_4 \sqrt{27}$

15. The pH of a solution is defined as $pH = -\log [H^+]$, where $[H^+]$ is the hydrogen ion concentration, in moles per litre (mol/L). How many times as acidic is the blueberry, with a pH of 3.2, as a saskatoon berry, with a pH of 4.0?

Saskatoon berries

Did You Know?

The saskatoon berry is a shrub native to Western Canada and the northern plains of the United States. It produces a dark purple, berry-like fruit. The Plains Cree called the berry “misaskwatomin,” meaning “fruit of the tree of many branches.” This berry is called “okonok” by the Blackfoot and “k’injíe” by the Dene tha’.

16. The apparent magnitude, m, of a celestial object is a measure of how bright it appears to an observer on Earth. The brighter the object, the lower the value of its magnitude. The difference between the apparent magnitudes, m_2 and m_1, of two celestial objects can be found using the equation $m_2 - m_1 = -2.5 \log \left(\frac{F_2}{F_1}\right)$, where F_1 and F_2 are measures of the brightness of the two celestial objects, in watts per square metre, and $m_2 < m_1$. The apparent magnitude of the Sun is -26.74 and the average apparent magnitude of the full moon is -12.74. How many times brighter does the sun appear than the full moon, to an observer on Earth?

17. The sound intensity, β, in decibels, is defined as $\beta = 10 \log \frac{I}{I_0}$, where I is the intensity of the sound, in watts per square metre (W/m^2), and I_0, the threshold of hearing, is 10^{-12} W/m^2. In some cities, police can issue a fine to the operator of a motorcycle when the sound while idling is 20 times as intense as the sound of an automobile. If the decibel level of an automobile is 80 dB, at what decibel level can police issue a fine to a motorcycle operator?

Device used to measure motorcycle noise levels

8.4 Logarithmic and Exponential Equations, pages 404–415

18. Determine the value of x, to two decimal places.

a) $3^{2x+1} = 75$

b) $7^{x+1} = 4^{2x-1}$

19. Determine x.

a) $2 \log_5 (x - 3) = \log_5 (4)$

b) $\log_4 (x + 2) - \log_4 (x - 4) = \frac{1}{2}$

c) $\log_2 (3x + 1) = 2 - \log_2 (x - 1)$

d) $\log \sqrt{x^2 - 21x} = 1$

20. A computer depreciates 32% per year. After how many years will a computer bought for \$1200 be worth less than \$100?

21. According to Kleiber's law, a mammal's resting metabolic rate, R, in kilocalories per day, is related to its mass, m, in kilograms, by the equation $\log R = \log 73.3 + 0.75 \log m$. Predict the mass of a wolf with a resting metabolic rate of 1050 kCal/day. Answer to the nearest kilogram.

22. Technetium-99m (Tc-99m) is the most widely used radioactive isotope for radiographic scanning. It is used to evaluate the medical condition of internal organs. It has a short half-life of only 6 h. A patient is administered an 800-MBq dose of Tc-99m. If none is lost from the body, when will the radioactivity of the Tc-99m in the patient's body be 600 MBq? Answer to the nearest tenth of an hour.

23. a) Mahal invests \$500 in an account with an annual percentage rate (APR) of 5%, compounded quarterly. How long will it take for Mahal's single investment to double in value?

b) Mahal invests \$500 at the end of every 3 months in an account with an APR of 4.8%, compounded quarterly. How long will it take for Mahal's investment to be worth \$100 000? Use the formula $FV = \frac{R[(1 + i)^n - 1]}{i}$, where FV is the future value, n is the number of equal periodic payments of R dollars, and i is the interest rate per compounding period expressed as a decimal.

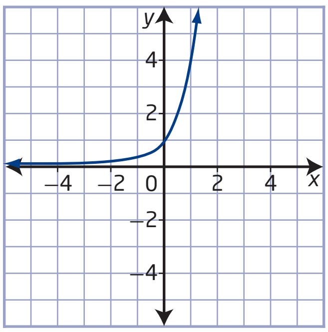
Chapter 8 Practice Test

Multiple Choice

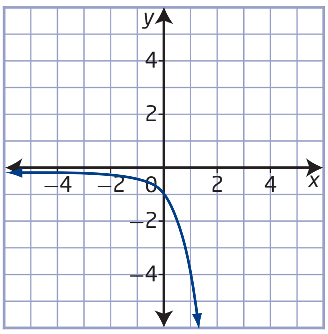
For #1 to #6, choose the best answer.

1. Which graph represents the inverse of $y = \left(\frac{1}{4}\right)^x$?

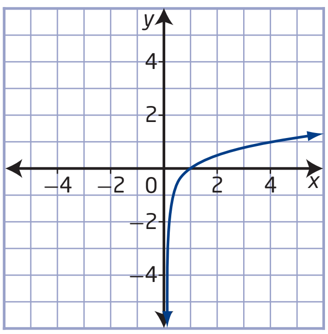
A



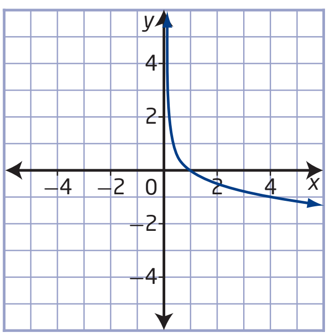
B



C



D



2. The exponential form of $k = -\log_h 5$ is

A $h^k = \frac{1}{5}$ **B** $h^k = -5$

C $k^h = \frac{1}{5}$ **D** $k^h = -5$

3. The effect on the graph of $y = \log_3 x$ if it is transformed to $y = \log_3 \sqrt{x + 7}$ can be described as

A a vertical stretch about the x-axis by a factor of $\frac{1}{2}$ and a vertical translation of 7 units up

B a vertical stretch about the x-axis by a factor of $\frac{1}{2}$ and a horizontal translation of 7 units left

C a horizontal stretch about the y-axis by a factor of $\frac{1}{2}$ and a vertical translation of 7 units up

D a horizontal stretch about the y-axis by a factor of $\frac{1}{2}$ and a horizontal translation of 7 units left

4. The logarithm $\log_3 \frac{x^p}{x^q}$ is equal to

A $(p - q) \log_3 x$ **B** $\frac{p}{q}$

C $p - q$ **D** $\frac{p}{q} \log_3 x$

5. If $x = \log_2 3$, then $\log_2 8\sqrt{3}$ can be represented as an algebraic expression, in terms of x, as

A $\frac{1}{2}x + 8$ **B** $2x + 8$

C $\frac{1}{2}x + 3$ **D** $2x + 3$

6. The pH of a solution is defined as $\text{pH} = -\log\,[H^+]$, where $[H^+]$ is the hydrogen ion concentration, in moles per litre (mol/L). Acetic acid has a pH of 2.9. Formic acid is 4 times as concentrated as acetic acid. What is the pH of formic acid?

A 1.1 **B** 2.3

C 3.5 **D** 6.9

Short Answer

7. Determine the value of x.

   a) $\log_9 x = -2$
   b) $\log_x 125 = \frac{3}{2}$
   c) $\log_3 (\log_x 125) = 1$
   d) $7^{\log_7 3} = x$
   e) $\log_2 8^{x-3} = 4$

8. If $5^{m+n} = 125$ and $\log_{m-n} 8 = 3$, determine the values of m and n.

9. Describe a series of transformations that could be applied to the graph of $y = \log_2 x$ to obtain the graph of $y = -5 \log_2 8(x - 1)$. What other series of transformations could be used?

10. Identity the following characteristics of the graph of the function $y = 2 \log_5 (x + 5) + 6$.

    a) the equation of the asymptote
    b) the domain and range
    c) the y-intercept
    d) the x-intercept

11. Determine the value of x.

    a) $\log_2 (x - 4) - \log_2 (x + 2) = 4$
    b) $\log_2 (x - 4) = 4 - \log_2 (x + 2)$
    c) $\log_2 (x^2 - 2x)^7 = 21$

12. Solve for x. Express answers to two decimal places.

    a) $3^{2x+1} = 75$
    b) $12^{x-2} = 3^{2x+1}$

Extended Response

13. Holly wins \$1 000 000 in a lottery and invests the entire amount in an annuity with an annual interest rate of 6%, compounded semi-annually. Holly plans to make a withdrawal of \$35 000 at the end of every 6 months. For how many years can she make the semi-annual withdrawals? Use the formula $PV = \frac{R[1 - (1 + i)^{-n}]}{i}$, where PV is the present value, n is the number of equal periodic payments of R dollars, and i is the interest rate per compounding period expressed as a decimal.

14. The exchange of free energy, ΔG, in calories (Cal), to transport a mole of a substance across a human cell wall is described as $\Delta G = 1427.6(\log C_2 - \log C_1)$, where C_1 is the concentration inside the cell and C_2 is the concentration outside the cell. If the exchange of free energy to transport a mole of glucose is 4200 Cal, how many times as great is the glucose concentration outside the cell as inside the cell?

15. The sound intensity, β, in decibels is defined as $\beta = 10 \log \frac{I}{I_0}$, where I is the intensity of the sound, in watts per square metre (W/m^2), and I_0, the threshold of hearing, is 10^{-12} W/m^2. A refrigerator in the kitchen of a restaurant has a decibel level of 45 dB. The owner would like to install a second such refrigerator so that the two run side by side. She is concerned that the noise of the two refrigerators will be too loud. Should she be concerned? Justify your answer.

16. Ethanol is a high-octane renewable fuel derived from crops such as corn and wheat. Through the process of fermentation, yeast cells duplicate in a bioreactor and convert carbohydrates into ethanol. Researchers start with a yeast-cell concentration of 4.0 g/L in a bioreactor. Eight hours later, the yeast-cell concentration is 12.8 g/L. What is the doubling time of the yeast cells, to the nearest tenth of an hour?

17. The Consumer Price Index (CPI) measures changes in consumer prices by comparing, through time, the cost of a fixed basket of commodities. The CPI compares prices in a given year to prices in 1992. The 1992 price of the basket is 100%. The 2006 price of the basket was 129.9%, that is, 129.9% of the 1992 price. If the CPI continues to grow at the same rate, in what year will the price of the basket be twice the 1992 price?

At the Movies

- Investigate one of your favourite movies. Find and record the box office revenues for the first 10 weeks. You may wish to change the time period depending on the availability of data, but try to get about ten successive data points.
- Graph the data.
- Which type of function do you think would best describe the graph? Is one function appropriate or do you think it is more appropriate to use different functions for different parts of the domain?
- Develop a function (or functions) to model the movie's cumulative box office revenue.
- Use your function to predict the cumulative revenue after week 15.
- Discuss whether this model will work for all movies.

Be prepared to present your findings to your classmates.

Cumulative Review, Chapters 7–8

Chapter 7 Exponential Functions

1. Consider the exponential functions $y = 4^x$ and $y = \frac{1}{4}^x$.

a) Sketch the graph of each function.

b) Compare the domain, range, intercepts, and equations of the asymptotes.

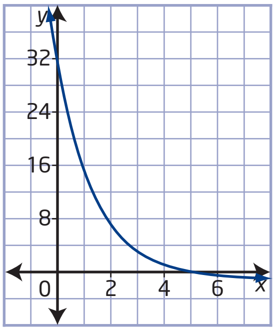
c) Is each function increasing or decreasing? Explain.

2. Match each equation with its graph.

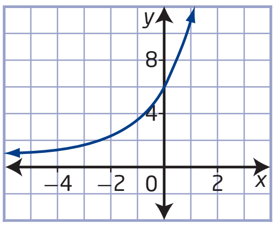
a) $y = 5(2^x) + 1$ **b)** $y = \left(\frac{1}{2}\right)^{x+5}$

c) $y + 1 = 2^{5-x}$ **d)** $y = 5\left(\frac{1}{2}\right)^{-x}$

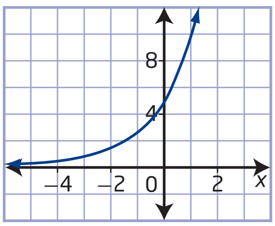
A



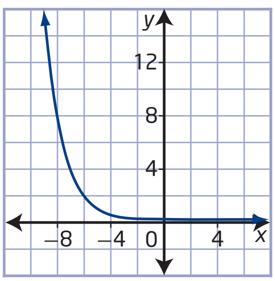
B



C



D



3. The number, B, of bacteria in a culture after t hours is given by $B(t) = 1000\left(2^{\frac{t}{3}}\right)$.

a) How many bacteria were there initially?

b) What is the doubling period, in hours?

c) How many bacteria are present after 24 h?

d) When will there be 128 000 bacteria?

4. The graph of $f(x) = 3^x$ is transformed to obtain the graph of $g(x) = 2(3^{x+4}) + 1$.

a) Describe the transformations.

b) Sketch the graph of $g(x)$.

c) Identify the changes in the domain, range, equations of the asymptotes, and any intercepts due to the transformations.

5. Write the expressions in each pair so that they have the same base.

a) 2^{3x+6} and 8^{x-5} **b)** 27^{4-x} and $\left(\frac{1}{9}\right)^{2x}$

6. Solve for x algebraically.

a) $5 = 2^{x+4} - 3$ **b)** $\frac{25^{x+3}}{625^{x-4}} = 125^{2x+7}$

7. Solve for x graphically. Round your answers to two decimal places.

a) $3(2^{x+1}) = 6^{-x}$

b) $4^{2x} = 3^{x-1} + 5$

8. A pump reduces the air pressure in a tank by 17% each second. Thus, the percent air pressure, p, is given by $p = 100(0.83^t)$, where t is the time, in seconds.

a) Determine the percent air pressure in the tank after 5 s.

b) When will the air pressure be 50% of the starting pressure?

Chapter 8 Logarithmic Functions

9. Express in logarithmic form.

a) $y = 3^x$ **b)** $m = 2^{a+1}$

10. Express in exponential form.

a) $\log_x 3 = 4$ **b)** $\log_a (x + 5) = b$

11. Evaluate.

a) $\log_3 \frac{1}{81}$

b) $\log_2 \sqrt{8} + \frac{1}{3} \log_2 512$

c) $\log_2 (\log_5 \sqrt{5})$

d) 7^k, where $k = \log_7 49$

12. Solve for x.

a) $\log_x 16 = 4$

b) $\log_2 x = 5$

c) $5^{\log_5 x} = \frac{1}{125}$

d) $\log_x (\log_3 \sqrt{27}) = \frac{1}{5}$

13. Describe how the graph of $y = \frac{\log_6 (2x - 8)}{3} + 5$ can be obtained by transforming the graph of $y = \log_6 x$.

14. Determine the equation of the transformed image of the logarithmic function $y = \log x$ after each set of transformations is applied.

a) a vertical stretch about the x-axis by a factor of 3 and a horizontal translation of 5 units left

b) a horizontal stretch about the y-axis by a factor of $\frac{1}{2}$, a reflection in the x-axis, and a vertical translation of 2 units down

15. The pH of a solution is defined as $\text{pH} = -\log [H^+]$, where $[H^+]$ is the hydrogen ion concentration, in moles per litre. The pH of a soil solution indicates the nutrients, such as nitrogen and potassium, that plants need in specific amounts to grow.

a) Alfalfa grows best in soils with a pH of 6.2 to 7.8. Determine the range of the concentration of hydrogen ions that is best for alfalfa.

b) When the pH of the soil solution is above 5.5, nitrogen is made available to plants. If the concentration of hydrogen ions is 3.0×10^{-6} mol/L, is nitrogen available?

16. Write each expression as a single logarithm in simplest form. State any restrictions on the variables.

a) $2 \log m - (\log \sqrt{n} + 3 \log p)$

b) $\frac{1}{3}(\log_a x - \log_a \sqrt{x}) + \log_a 3x^2$

c) $2 \log (x + 1) + \log (x - 1) - \log (x^2 - 1)$

d) $\log_2 27^x - \log_2 3^x$

17. Zack attempts to solve a logarithmic equation as shown. Identify and describe any errors, and then correctly solve the equation.

$\log_3 (x - 4)^2 = 4$

$3^4 = (x - 4)^2$

$81 = x^2 - 8x + 16$

$0 = x^2 - 8x - 65$

$x = -13$ or $x = 5$

18. Determine the value of x. Round your answers to two decimal places if necessary.

a) $4^{2x+1} = 9(4^{1-x})$

b) $\log_3 x + 3 \log_3 x^2 = 14$

c) $\log (2x - 3) = \log (4x - 3) - \log x$

d) $\log_2 x + \log_2 (x + 6) = 4$

19. The Richter magnitude, M, of an earthquake is related to the energy, E, in joules, released by the earthquake according to the equation $\log E = 4.4 + 1.4M$.

a) Determine the energy for earthquakes with magnitudes 4 and 5.

b) For each increase in M of 1, by what factor does E change?

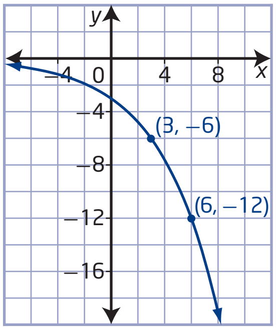
20. At the end of each quarter year, Aaron makes a \$625 payment into a mutual fund that earns an annual percentage rate of 6%, compounded quarterly. The future value, FV, of Aaron's investment is $FV = \frac{R[(1 + i)^n - 1]}{i}$, where n is the number of equal periodic payments of R dollars, and i is the interest rate per compounding period expressed as a decimal. After how long will Aaron's investment be worth \$1 000 000?

Unit 3 Test

Multiple Choice

For #1 to #7, select the best answer.

1. The graph of the function $y = a(2^{bx})$ is shown.



The value of a is

A 3

B $\frac{1}{3}$

C $-\frac{1}{3}$

D -3

2. The graph of the function $y = b^x$, $b > 1$, is transformed to $y = 3(b^{x+1}) - 2$. The characteristics of the function that change are

A the domain and the range

B the range, the x-intercept, and the y-intercept

C the domain, the x-intercept, and the y-intercept

D the domain, the range, the x-intercept, and the y-intercept

3. The half-life of carbon-14 is 5730 years. If a bone has lost 40% of its carbon-14, then an equation that can be used to determine its age is

A $60 = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

B $60 = 100\left(\frac{1}{2}\right)^{\frac{5730}{t}}$

C $40 = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

D $40 = 100\left(\frac{1}{2}\right)^{\frac{5730}{t}}$

4. Which of the following is an equivalent form for $2x = \log_3 (y - 1)$?

A $y = 3^{2x} - 1$

B $y = 3^{2x+1}$

C $y = 9^x + 1$

D $y = 9^{x+1}$

5. The domain of $f(x) = -\log_2 (x + 3)$ is

A $\{x \mid x > -3, x \in \mathrm{R}\}$

B $\{x \mid x \geq -3, x \in \mathrm{R}\}$

C $\{x \mid x < 3, x \in \mathrm{R}\}$

D $\{x \mid x \in \mathrm{R}\}$

6. If $\log_2 5 = x$, then $\log_2 \sqrt[4]{25^3}$ is equivalent to

A $\frac{3x}{2}$

B $\frac{3x}{8}$

C $x^{\frac{3}{2}}$

D $x^{\frac{3}{8}}$

7. If $\log_4 16 = x + 2y$ and $\log 0.0001 = x - y$, then the value of y is

A -2

B $-\frac{1}{2}$

C $\frac{1}{2}$

D 2

Numerical Response

Copy and complete the statements in #8 to #12.

8. The graph of the function $f(x) = \left(\frac{1}{4}\right)^x$ is transformed by a vertical stretch about the x-axis by a factor of 2, a reflection about the x-axis, and a horizontal translation of 3 units right. The equation of the transformed function is ■.

9. The quotient $\dfrac{9^{\frac{1}{2}}}{27^{\frac{2}{3}}}$ expressed as a single power of 3 is ■.

10. The point P(2, 1) is on the graph of the logarithmic function $y = \log_2 x$. When the function is reflected in the x-axis and translated 1 unit down, the coordinates of the image of P are ■.

11. The solution to the equation $\log 10^x = 0.001$ is ■.

12. Evaluating $\log_5 40 - 3 \log_5 10$ results in ■.

Written Response

13. Consider $f(x) = 3^{-x} - 2$.

a) Sketch the graph of the function.

b) State the domain and the range.

c) Determine the zeros of $f(x)$, to one decimal place.

14. Solve for x and verify your solution.

a) $9^{\frac{1}{4}}\left(\frac{1}{3}\right)^{\frac{x}{2}} = \sqrt[3]{27^4}$

b) $5(2^{x-1}) = 10^{2x-3}$

15. Let $f(x) = 1 - \log (x - 2)$.

a) Determine the domain, range, and equations of the asymptotes of $f(x)$.

b) Determine the equation of $f^{-1}(x)$.

c) Determine the y-intercepts of $f^{-1}(x)$.

16. Solve for x algebraically.

a) $\log 4 = \log x + \log (13 - 3x)$

b) $\log_3 (3x + 6) - \log_3 (x - 4) = 2$

17. The following shows how Giovanni attempted to solve the equation $2(3^x) = 8$. Identify, describe, and correct his errors.

$2(3^x) = 8$

$6^x = 8$

$\log 6^x = \log 8$

$x \log 6 = \log 8$

$x = \dfrac{\log 8}{\log 6}$

$x = \log 8 - \log 6$

$x \approx 0.12$

The solution is $x \approx 0.12$.

18. The Richter magnitude, M, of an earthquake is defined as $M = \log\left(\frac{A}{A_0}\right)$, where A is the amplitude of the ground motion and A_0 is the amplitude, corrected for the distance to the actual earthquake, that would be expected for a standard earthquake. An earthquake near Tofino, British Columbia, measures 5.6 on the Richter scale. An aftershock is $\frac{1}{4}$ the amplitude of the original earthquake. Determine the magnitude of the aftershock on the Richter scale, to the nearest tenth.

19. The world population was approximately 6 billion in 2000. Assume that the population grows at a rate of 1.3% per year.

a) Write an equation to represent the population of the world.

b) When will the population reach at least 10 billion?

20. To save for a new highway tractor, a truck company deposits \$11 500 at the end of every 6 months into an account with an annual percentage rate of 5%, compounded semi-annually. Determine the number of deposits needed so that the account has at least \$150 000. Use the formula $FV = \dfrac{R[(1 + i)^n - 1]}{i}$, where FV is the future value, n is the number of equal periodic payments of R dollars, and i is the interest rate per compounding period expressed as a decimal.