**Chapter - Exponential Functions, Logarithms and Geometric Series**

**I – Exponential Functions**

A function is an exponential function if the variable is in an exponent.  
  
Examples : ,   
  
Basic exponential functions (without transformations) are in the form :  
  
 where *x* is the variable as an exponent and *c* is a positive base

Exemples : a) b)

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *x* | -2 | -1 | 0 | 1 | 2 | 3 |
| *y* |  |  |  |  |  |  |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *x* | -2 | -1 | 0 | 1 | 2 | 3 |
| *y* |  |  |  |  |  |  |

We notice that on both graphs:

* Domain =
* Range =
* Horizontal asymptote : .
* The y-intercept is always 1
* No x-intercepts
* We can read the value of the base when *x* = 1.   
  We know the reciprocal of the base when *x* = -1.

This is true for all basic exponential functions (without transformations).

Furthermore :

If the base the function is increasing.  
   
  
 If the base the function is decreasing.  
   
  
 If the base the function is constant : .

   
  
Hwk : p 342 # 1 – 6, 8 – 10, 12

**II – Logarithmic Functions**

1. **Definition of a Logarithm**  
     
   The inverse of exponential functions are called logarithmic functions.  
     
    is the inverse of

The inverse functions allow us to express a relation the “other way”.  
Indeed, the functions and are inverses of each other when .  
 and can be expressed as .  
  
In the same way, can be written as   
  
We notice that the result of a logarithm is the exponent of the exponential…  
  
Examples :   
  
 can be written as :   
  
 can be written as :  
  
 can be written as :  
  
 can be written as :  
  
 can be written as :  
  
 can be written as:  
  
 can be written as :  
  
**Note :** We can change the form (exp 🡪 log ou log 🡪 exp) only when the exp. function or the log ISOLATED!

Furthermore, since inverse functions cancel each other’s actions, we get :

et.

Examples :

Applications : Determine the exact values of the following :

Note : If there is no written base, we assume that the base is 10…   
  
Notice: if the argument and the base are powers of the same number, we can evaluate the value of the logarithm without using a calculator.

Your turn p 374

1. **Basic Logarithmic Functions (non-transformed)**  
     
   To represent a logarithmic function graphically, we can either make a table of values of the corresponding exponential function and then exchange their coordinates **or** use the symmetry of the inverse by using the linear equation (like we learned in chapter 1).   
     
   **ATTENTION**: Exponential functions do not satisfy the horizontal line test when their base is not 1. This means that logarithmic functions with a base 1 are not defined!!!!  
     
   Furthermore, by noticing the properties of the inverse function, we can tell that the domain and image are switched. Therefore :

Examples : a)

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *x* |  |  |  |  |  |  |
| *y* |  |  |  |  |  |  |



b)

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *x* |  |  |  |  |  |  |
| *y* |  |  |  |  |  |  |



We notice that for both graphs :

* Vertical Asymptote : .
* No y-intercept (not on the domain)
* x-intercept pour .
* We can read the value of the base when *y* = 1.

This is true for all the basic logarithmic functions (non-transformed).

Furthermore :

If the base the function is increasing.  
   
 If the base the function is decreasing.  
   
 If the base the function is not defined!  
  
**Note :** With a calculator :   
  
Hwk : p 380 # 1 – 6, 8, 9, 12 – 16, 21

1. **Exponentials and logarithms with base e**  
   Of all the famous irrational numbers (like ), there is one that is very important to calculus: the number .  
     
   The logarithm with base e is known as the natural logarithm. We note it as : *ln*.

**III – Transformation of Exponential and Logarithmic Functions**

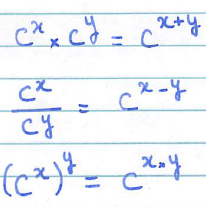
Where *a* is a vertical stretch or a reflection around the x-axis, *b* is a horizontal stretch or a reflection around the y-axis, *h* is a horizontal translation and *k* is a vertical translation

It is important to remember that exponential functions have a horizontal asymptote which will only “move” if we apply a vertical translation.  
  
Logarithmic functions have a vertical asymptote which will only “move” if we apply a horizontal translation.

Examples :   
i)   
  


ii)   
  
  
  
iii)   
  
  
  
Hwk : p 354 # 4, 6, 7, 9, 11, 12 & p 389 # 2, 4 – 6, 9, 10, 12 – 14, 17.

**IV – Laws of exponents and logarithms**

1. **Reminder on exponent laws**  
     
     
     
   Examples : Simplify



6. **Logarithm Laws**

(n, x, y and c are positive, )

Examples :   
i)   
  
ii)   
  
iii)   
  
iv)   
  
Your Turn p 395  
  
v)   
  
vi)   
  
vii)

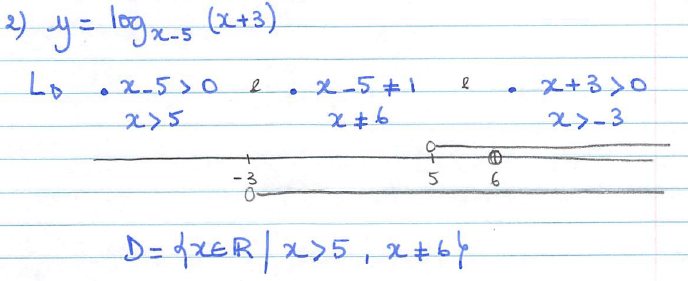
Your Turn p 396  
viii)   
  
  
  
ix)

Hwk : p 400 # 1 – 3, 5 – 13, 16, 18, 20.

1. **Restrictions on the domain for logarithms**  
     
   Reminder of the restrictions: existes if

Examples: Determine the restrictions on the domain of:

1)

  
Your Turn:

Restriction are REQUIRED when we solve an equation.   
  
ATTENTION, it is important to determine the restrictions before simplifying !!

Examples :   
a)

b)

Hwk : Extra Practice

**V – Exponential and logarithmic equations**

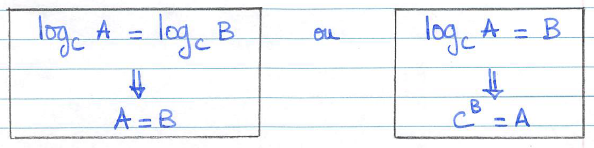
1. **Solve Graphically**  
     
   To Solve any equation graphically, you need to represent y each side of the equation graphically and find the x coordinate of their intersection, or put all the terms on the same side and find the zero of the given function.  
     
   Example : Solve or   
     
       
     
     
   We can go faster and use a graphic calculator if we do not need to show our work…
2. **Algebraic methods for exponential equations**   
      
   We notice that exponential equations **do not have any restriction**!

* **Write all the terms on the same base (FIRST method to try)**   
    
  Examples : Write all the terms on base 3  
  i) 27  
    
  ii)   
    
  iii)

Your Turn p 360

Applications : Solve :  
  
i)   
  
  
  
  
  
  
  
ii)   
  
  
  
  
  
  
  
  
 Your Turn p 361  
Attention, this is not always possible!  
  
Hwk : p 364 # 1 – 5, 7, 16, 17

* **Use logarithms (if we cannot write the terms on the same base)**  
    
  Examples :   
  i)   
    
    
    
    
    
    
    
  ii)   
    
    
    
    
    
    
    
    
    
  iii)   
    
    
    
    
    
    
    
  Hwk : p 412 # 2, 7

1. **Algebraic methods for Logarithmic equations**  
     
   Like all equations, we need to start by determining the restriction. Logarithms have restrictions…  
     
   To Solve a logarithmic equation, we need to isolate the logarithmic function in order to transform it into its exponential equivalent (to “free” the variables from the logarithm)…  
     
   To do this, we are able to use the logarithm laws to assure that there is only one logarithm on each side.   
     
     
     
   Examples :   
   i)   
     
     
     
     
     
     
     
   ii)   
     
     
     
     
     
     
     
     
     
   iii)   
     
     
     
     
     
     
     
     
     
     
     
     
     
     
     
     
     
     
     
     
   Notice : Any number can be written as a logarithme (on any base )   
     
   ex :   
     
   Hwk : p 412 # 3, 5, 6, 8
2. **Solving Classic Problems**  
     
   Exponential function are occasionally used to solve “real life” situations like: radioactive evolution, evolution of a population and bacteria growth.

Where *A* is the quantity at time t, *P* is the ignition quantity (qd *t*=0), *r* is the rate of evolution, *t* is the time, and *n* is the frequency that the rate of evolution is applied (same unites as *t*).

Examples:

* + - A population of bacteria triples every 10 days. You start with 5 bacteria. How many bacteria are there after 55 days?

bacteria

* + - The radioactive element Americium has a half life of 432 years. If you start with 10mg of Americium, how much will remain after t years

t: # of years A: remaining quantity of Americium in mg

* + - The population of a city increases by 5% each year. Today, there are 20 000 people. How many will there be in t years?

t: number of years P: population

Reminder:

increases by 12% : r=1.12

decreases by 7% : r=0.93

is 20% of : r=0.2

* + - 1500 dollars is deposited in a bank with an interest of 4.3% compounded quarterly. What is the value of the placement at the start of 6 years.

Reminder: if the interest is compounded semi-annually, you obtain half of the interest two times per year.

if the interest is compounded monthly, you obtain one twelfth of the interest twelve times per year.

Another classic example : Earthquakes (or sound levels):  
  
The magnitude, M (on the Richter scale) and the amplitude, A (the force) of an earthquake are connected by the relation : , where is a constant.  
  
If the first earthquake has a magnitude of 6.4, and a second earthquake is 15 times more powerful, what is the magnitude of the second earthquake?

Hwk : p 364 # 9 – 14 p 381 # 19 p 401 # 13, 16 p 412 # 10 – 13, 15 – 18.

**VI – Geometric Sequences and Series**

1. **Sequences**  
     
   A **geometric Sequence** is a list of values, where each value is obtained by multiplying the previous value by the same number (which we call the reason of the suite) .  
     
   Exemples : 2, 6, 18, 54, …  
    1, -5, 25, -125, …  
    18, 6, 2, , , , …  
     
   To determine the reason of a geometric sequence (if it is not evident), we must divide one term by the previous term.  
     
   To tell is the sequence is geometric, we must divide each term by the previous term and obtain the same value each time.   
     
   **General Term** of a geometric sequence : (it’s a formula which will give us any term if we know its position)

Where is the term, is the value of the first term, *r* is the reason and *n* is the position of the term .  
   
  
Examples : with the 3 following examples, we obtain  
   
  
   
   
   
  
We recognize that a geometric sequence is obtained if we restrict the domain of an exponential function to only natural numbers… (this is why geometric sequences are in this chapter).  
  
  
Applications :   
i) **Determine the general term**  
 Bacteria reproduces by dividing themselves into two. If you have a sample of 10 bacteria, express the number of bacteria you will have in the nth generation.  
  
  
  
  
ii) **Determining a specific term**  
 A photocopier allows you to reduce the image to 67% of their original size. If we reduce a 25cm square photo 5 times in a row, what size will the image be?  
  
  
  
  
  
  
iii) **Determining t1 and r**  
 The third term of a geometric sequence is 54 and its sixth term is -1458. Determine the values of the three first terms in the sequence.

iv) **Determining the number of terms.**  
 Determine the number of terms in the geometric sequence : 2, 1, , …,   
  
  
  
  
  
  
  
  
Hwk : worksheet p 39 # 1, 2, 3ab, 4 – 6, 8 – 10, 12, 14 – 16, 20, 25 + supplementary exercises

1. **Series and Sigma Notation**

A **geometric serie** is the somme of the terms of the geometric sequence.

Generally, we note Sn as the general term of a geometric series. Sn = t1 + t2 + … + tn.

Example : The row 5, 15, 45, 135, … is a geometric sequence with a reason of 3 and its first term is 5.   
 We have : S4 = 5 + 15 + 45 + 135 = 200.

General Formulas for geometric series :

* This formula is used if we do not know the number of terms nor the last term…

This formula is used if we know the last term but not the number of terms…  
  
Examples :   
i) Determine the sum of the first 10 terms of each series :  
 et   
  
  
  
  
  
  
ii) Determine the sum of the terms of each series :  
   
  
  
  
  
  
  
  
  
Exercises p 53 # 2ac, 3ac, 4ac, 5 – 7, 10, 13, 14, 16, 18, 19, 22.

**Sigma Notation :**   
When we want to add many values that depend on a variable that takes all the integer values of an internal, we use the Sigma notation .

Examples :

7. **Infinite Series**   
   An infinite sequence is a sequence that does not have a last term.   
   An infinite series is an infinite sum of terms…

Definitions :

* We say that a **series** is **convergente** if Sn tends to a fixed value when n becomes very large…  
    
  Example :   
   When we keep adding terms, the sum approaches the value 2.   
   We note S∞ = 2
* We say that a series is divergent if Sn does not tend to a particular value when n becomes large…  
    
  Example :   
   The more numbers we add, the more the sum approaches infinity…  
   We cannot use the notation S∞!

Properties : The geometric series with ratios stricly between -1 and 1 are convergent.   
All others are divergent.

**Convergent series iff**

General formula to calculate the infinite sum of a convergent geometric series :

S∞ =

Examples :   
a) Determine if each series is convergent or divergent and calculate its sum if possible :  
   
  
  
  
b) For which values of the variable are the following sequences convergent ?

Hwk : worksheet p 63 # 1 – 5, 6 – 12, 15 – 17, 19, 22