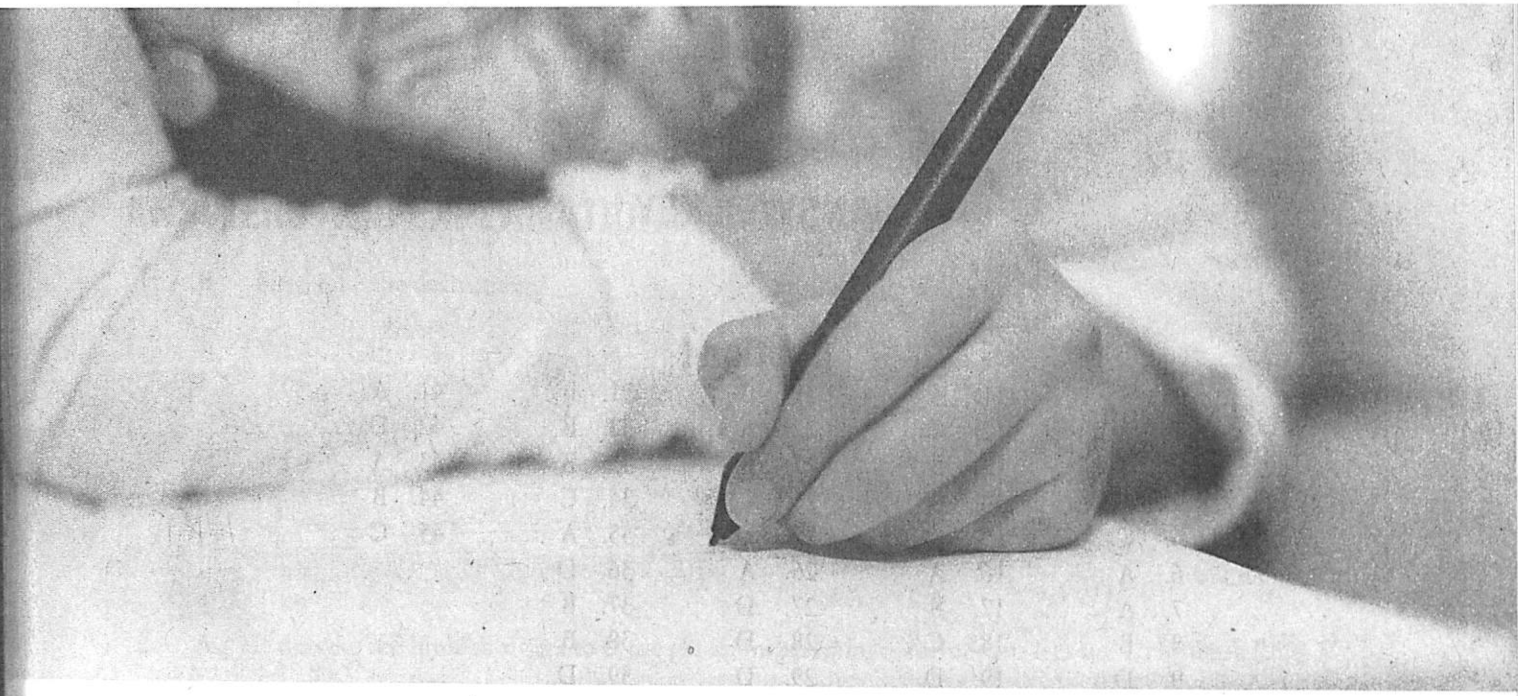


Practice Test 2: Answers and Explanations

ANSWER KEY

Section I

1. B	11. A	21. B	31. B	41. A
2. A	12. C	22. A	32. B	42. D
3. C	13. B	23. B	33. B	43. A
4. D	14. B	24. C	34. C	44. B
5. C	15. D	25. D	35. A	45. C
6. A	16. A	26. A	36. D	
7. A	17. B	27. D	37. B	
8. B	18. C	28. D	38. B	
9. D	19. D	29. D	39. D	
10. D	20. D	30. C	40. D	

ANSWERS AND EXPLANATIONS TO SECTION I

1. **B** First, take the derivative.

$$g'(x) = \frac{1}{32}\left(4x^3\right) - 5(2x) = \frac{x^3}{8} - 10x$$

Now, plug in 4 for x.

$$\frac{(4)^3}{8} - 10(4) = 8 - 40 = -32$$

2. **A** If we take the limit as x goes to 0, we get an indeterminate form $\frac{0}{0}$, so let's use L'Hôpital's Rule. We take the derivative of the numerator and the denominator and we get $\lim_{x \to 0} \frac{8x^2}{\cos x - 1} = \lim_{x \to 0} \frac{16x}{(-\sin x)}$. When we take the limit, we again get an indeterminate form $\frac{0}{0}$, so let's use L'Hôpital's Rule a second time. We take the derivative of the numerator and the denominator and we get $\lim_{x \to 0} \frac{16x}{(-\sin x)} = \lim_{x \to 0} \frac{16}{(-\cos x)}$. Now, when we take the limit we get: $\lim_{x \to 0} \frac{16}{(-\cos x)} = -16$.

3. **C** Notice that if we plug 5 into the expressions in the numerator and the denominator, we get $\frac{0}{0}$, which is undefined. Before we give up, we need to see if we can simplify the limit so that it can be evaluated. If we factor the expression in the numerator, we get $\frac{(x+5)(x-5)}{(x-5)}$, which can be simplified to $x + 5$. Now, if we take the limit (by plugging in 5 for x), we get 10.

4. **D** We need to use the Quotient Rule, which is Given $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$.

Here we have

$$f'(x) = \frac{(x^3 + 7)(5x^4 - 1) - (x^5 - x + 2)(3x^2)}{(x^3 + 7)^2}$$

5. **C** There are two ways we could evaluate this limit. First, we could recognize that this limit is in the form of the Definition of the Derivative $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Then the limit is the derivative of $f(x) = \tan x$ at $x = \frac{\pi}{4}$. The derivative of $f(x) = \tan x$ is $f'(x) = \sec^2 x$, and at $x = \frac{\pi}{4}$, we get $f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2$. Second, we have a limit of the indeterminate form $\frac{0}{0}$, so we can use L'Hôpital's Rule to find the limit. Take the derivative of the numerator and the denominator: $\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - 1}{h} = \lim_{h \to 0} \frac{\sec^2\left(\frac{\pi}{4} + h\right)}{1}$. Now we take the limit to get $\sec^2\left(\frac{\pi}{4}\right) = 2$.

6. **A** First, rewrite the integral as $\int x\left(\sqrt{3}\right)x^{\frac{1}{2}}\,dx$.

Now, we can simplify the integral to $\sqrt{3}\int x^{\frac{3}{2}}\,dx$.

Next, use the power rule for integrals, which is $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$.

Then, we get $\sqrt{3}\int x^{\frac{3}{2}}\,dx = \sqrt{3}\,\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C = \frac{2\sqrt{3}}{5}x^{\frac{5}{2}} + C$.

7. **A** The simplest thing to do here is to find $\lim_{x \to 1}$ of both pieces of the function and set them equal to each other. We can do this by plugging $x = 1$ into both pieces: $1 - 3k + 2 = 5 - k$. If we solve for k, we get $k = -1$.

8. **B** We have to find the areas of the regions between the curve and the x-axis on the interval from $x = 0$ to $x = 5$. The regions are all triangles, so we can easily find the areas. Note that the area of the region from $x = 2$ to $x = 4$ is negative. We get: $\frac{1}{2}(2)(2) - \frac{1}{2}(2)(4) + \frac{1}{2}(1)(2) = -1$.

9. **D** Use the Chain Rule: $\frac{dy}{dx} = \sec(\pi x^2)\tan(\pi x^2)(2\pi x)$.

10. **D** First, take the derivative: $\frac{dy}{dx} = -\frac{2}{3}(x-2)^{-\frac{1}{3}}$, which can be rewritten as $\frac{dy}{dx} = -\frac{2}{3\sqrt[3]{x-2}}$. If we plug in $x = 2$, we get zero in the denominator, so the derivative does not exist at $x = 2$.

11. **A** This integral is of the form $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$, where $a = 1$.

Thus, we get

$$\int_0^{\frac{1}{2}} \frac{2dx}{\sqrt{1-x^2}} = 2\sin^{-1}(x)\Big|_0^{\frac{1}{2}} = 2\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right] = 2\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{3}$$

12. **C** In order for f to be continuous, three conditions must be met. First, $f(2)$ must exist: $f(2) = 4 - 6b + 2 = 6 - 6b$. Next, $\lim_{x\to 2} f(x)$ must exist. The limit from the left is $\lim_{x\to 2^-} f(x) = 6 - 6b$, and the limit from the right is $\lim_{x\to 2^+} f(x) = 8b - 8$. We set the two limits equal to each other and solve for b: $6 - 6b = 8b - 8$. So $b = 1$. The final condition is that $\lim_{x\to 2} f(x) = f(2)$. At $b = 1$, $\lim_{x\to 2} f(x) = f(2) = 0$. Therefore, b equals 1, and the answer is (C), 1.

13. **B** First, we need to take the derivative of f using the Product Rule: $f'(x) = e^{-x} - xe^{-x}$.

Next, we set it equal to zero and solve: $e^{-x} - xe^{-x} = 0$. Factor: $e^{-x}(1 - x) = 0$. Because e^{-x} is always positive, the only solution is $x = 1$. Next, we need to verify that $x = 1$ is a maximum. Plug in values less than and greater than 1 to make sure that the derivative changes sign there: $f'(0) = e^0 - (0)e^0 > 0$ and $f'(2) = e^{-2} - (2)e^{-2} < 0$. So, $x = 1$ is the x-coordinate of the maximum. The maximum value occurs at the y-coordinate, which we find by plugging $x = 1$ into the original equation: $f(1) = (1)e^{-1} = \frac{1}{e}$.

14. **B** We can use implicit differentiation to find $\frac{dy}{dx}$. First, differentiate with respect to x: $3y^2\frac{dy}{dx} = x\frac{dy}{dx} + y - 4x$. Next, plug in (1, 2) for x and y: $3(2)^2\frac{dy}{dx} = (1)\frac{dy}{dx} + (2) - 4(1)$. Simplify: $12\frac{dy}{dx} = \frac{dy}{dx} - 2$. And solve for $\frac{dy}{dx}$ on the left and the terms without $\frac{dy}{dx}$ on the right: $\frac{dy}{dx} = -\frac{2}{11}$.

15. **D** If we take the limit as x goes to 0, we get an indeterminate form $\frac{0}{0}$, so let's use L'Hôpital's Rule. We take the derivative of the numerator and the denominator, and we get $\lim_{x\to 0}\frac{x\cdot 2^x}{2^x - 1} = \lim_{x\to 0}\frac{x\cdot 2^x \ln 2 + 2^x}{2^x \ln 2}$.

Now, when we take the limit, we get $\lim_{x\to 0}\frac{x\cdot 2^x \ln 2 + 2^x}{2^x \ln 2} = \frac{1}{\ln 2}$.

16. **A** We can evaluate this integral using u-substitution. Let $u = 1 + x^2$ and $du = 2xdx$, so $\frac{1}{2}du = xdx$. Substitute into the integrand: $\int x\sec^2(1 + x^2)dx = \frac{1}{2}\int \sec^2 u\, du$. Integrate: $\frac{1}{2}\int \sec^2 u\, du = \frac{1}{2}\tan u + C$ and substitute back: $\frac{1}{2}\tan u + C = \frac{1}{2}\tan(1 + x^2) + C$.

17. **B** First, we need to find $\frac{dy}{dx}$. It's simplest to find it implicitly.

$$18x + 32y\frac{dy}{dx} = 0$$

Now, solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{18x}{32y} = -\frac{9x}{16y}$$

Next, plug in $x = 2$ and $y = -1$ to get the slope of the tangent line at the point.

$$\frac{dy}{dx} = \frac{-18}{-16} = \frac{9}{8}$$

Now, use the point-slope formula to find the equation of the tangent line.

$$(y + 1) = \frac{9}{8}(x - 2)$$

If we multiply through by 8, we get $8y + 8 = 9x - 18$ or $9x - 8y - 26 = 0$.

18. **C** First, we need to check the sign of the particle's velocity in the time interval. $v(t) = 12t - 3t^2 = 3t(4 - t) = 0$ at $t = 0$ and $t = 4$. Let's pick a number between 0 and 4, like 1: $v(1) = 12(1) - 3(1)^2 > 0$. Next, we pick a number greater than 4, like 5: $v(5) = 12(5) - 3(5)^2 < 0$. This means that the particle is moving to the right for the first four seconds and to the left for the last two seconds. We get the distance traveled in the first interval with $\int_0^4 12x - 3x^2\,dx = \left(6x^2 - x^3\right)\Big|_0^4 = 32 - 0 = 32$. We get the distance traveled in the second interval with $-\int_4^6 12x - 3x^2\,dx = -\left(6x^2 - x^3\right)\Big|_4^6 = -(0 - 32) = 32$. Therefore, the distance that the particle travels is $32 + 32 = 64$.

19. **D** The derivative of an expression of the form a^u, where u is a function of x, is

$$\frac{d}{dx}a^u = a^u(\ln a)\frac{du}{dx}$$

Here we get

$$\frac{d}{dx}3^{\pi x} = 3^{\pi x}(\ln 3)\pi$$

20. **D** In order to find the average value, we use the Mean Value Theorem for Integrals, which says that the average value of $f(x)$ on the interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Here we have $\frac{1}{e-1}\int_1^e \frac{1}{x}\,dx$.

Evaluating the integral, we get $\ln x\Big|_1^e = \ln e - \ln 1 = 1$. Therefore, the answer is $\frac{1}{e-1}$.

21. **B** Use the Chain Rule to find the derivative: $\frac{dy}{dx} = 6\left(x^4 + \sin x\right)^5\left(4x^3 + \cos x\right)$.

22. **A** First, we need to find $\frac{dy}{dx}$ using implicit differentiation.

$$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y\right)\sin xy$$

Rather than simplifying this, simply plug in (0, 1) to find $\frac{dy}{dx}$.

We get $\frac{dy}{dx} = 1$.

This means that the slope of the tangent line at (0, 1) is 1, so the slope of the normal line at (0, 1) is the negative reciprocal, which is –1.

23. **B** We can evaluate this integral using *u*-substitution. Let $u = \sqrt{x}$ and $du = \frac{1}{2\sqrt{x}}dx$, so $2du = \frac{1}{\sqrt{x}}dx$.

Substituting into the integrand, we get $2\int \csc^2 u\,du = -2\cot u + C$. Now we substitute back: $-2\cot u + C = -2\cot\sqrt{x} + C$.

24. **C** We will need to use the fact that $\lim_{x\to 0}\frac{\sin x}{x} = 1$ to find the limit.

First, rewrite the limit as

$$\lim_{x\to 0}\frac{\sin^3(2x)}{x^3\cos^3(2x)}$$

Next, break the fraction into

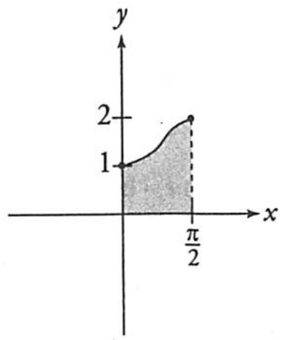
$$\lim_{x\to 0}\left(\frac{\sin^3(2x)}{x^3}\,\frac{1}{\cos^3(2x)}\right)$$

Now, if we multiply the top and bottom of the first fraction by 8, we get

$$\lim_{x\to 0}\frac{8\sin^3(2x)}{(2x)^3}\,\frac{1}{\cos^3(2x)}$$

Now, we can take the limit, which gives us $8(1)(1) = 8$.

25. **D** First, let's graph the curve.



We can find the volume by taking a vertical slice of the region. The formula for the volume of a solid of revolution around the x-axis, using a vertical slice bounded from above by the curve $f(x)$ and from below by $g(x)$, on the interval $[a, b]$ is

$$\pi\int_a^b\left[f(x)^2 - g(x)^2\right]dx$$

Here we get

$$\pi\int_0^{\frac{\pi}{2}}(1 + \sin^2 x)^2\,dx$$

26. **A** We use the Chain Rule and the Quotient Rule.

$$\frac{dy}{dx} = 4\left(\frac{x^3 - 2}{2x^5 - 1}\right)^3\left[\frac{(2x^5 - 1)(3x^2) - (x^3 - 2)(10x^4)}{(2x^5 - 1)^2}\right]$$

If we plug in 1 for x, we get

$$\frac{dy}{dx} = 4(-1)^3\left[\frac{3 + 10}{1^2}\right] = -52$$

27. **D** We can evaluate this integral using u-substitution. Let $u = 5 - x$ and $5 - u = x$. Then $-du = dx$.

Substituting, we get

$$-\int(5 - u)u^{\frac{1}{2}}\,du$$

The integral can be rewritten as

$$-\int\left(5u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

Evaluating the integral, we get

$$-5\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+\frac{u^{\frac{5}{2}}}{\frac{5}{2}}+C$$

This can be simplified to

$$-\frac{10}{3}u^{\frac{3}{2}}+\frac{2}{5}u^{\frac{5}{2}}+C$$

Finally, substituting back, we get

$$-\frac{10}{3}(5-x)^{\frac{3}{2}}+\frac{2}{5}(5-x)^{\frac{5}{2}}+C$$

28. **D** First, separate the variables: $\frac{dy}{y}=-2dt$. Integrate both sides: $\int\frac{dy}{y}=-2\int dt$, $\ln y=-2t+C$. We can isolate y by exponentiating both sides: $y=e^{-2t+C}=Ae^{-2t}$. Now, plug in the initial condition to solve for the constant: $100=Ae^0=A$. We get $y=100e^{-2t}$. Therefore, $y(2)=100e^{-2(2)}=100e^{-4}=\frac{100}{e^4}$.

29. **D** In order to find possible points of inflection, we first need to find the second derivative:

$$\frac{dy}{dx}=4x^3-18x^2-48x$$
$$\frac{d^2y}{dx^2}=12x^2-36x-48$$

Next, we set the second derivative equal to zero and solve: $12x^2-36x-48=0$. Divide through by 12: $x^2-3x-4=0$, which factors to $(x-4)(x+1)=0$. We get $x=4$ and $x=-1$. In order to be sure that $x=4$ is a point of inflection, we need to check that the second derivative changes sign there. Plug in values less than and great than 4 and check the sign of the second derivative: $12(3)^2-36(3)-48<0$ and $12(5)^2-36(5)-48>0$, so $x=4$ is a point of inflection.

30. **C** First, rewrite the integral as $\int_0^1\frac{\sin x}{\cos x}\,dx$.

Now, we can use u-substitution to evaluate the integral. Let $u=\cos x$. Then $du=-\sin x$. We can also change the limits of integration. The lower limit becomes $\cos 0=1$ and the upper limit becomes $\cos 1$, which we leave alone. Now we perform the substitution, and we get

$$-\int_1^{\cos 1}\frac{du}{u}$$

Evaluating the integral, we get $-\ln u\Big|_1^{\cos 1}=-\ln(\cos 1)+\ln 1=-\ln(\cos 1)$. This log is also equal to $\ln(\sec 1)$.

31. **B** We can find the slope of the tangent line by taking the derivative: $\frac{dy}{dx} = -\sin x$. Next, we need to find where the graph has a zero: $\frac{1}{2} + \cos x = 0$. $\cos x = -\frac{1}{2}$ in the interval at $x = \frac{2\pi}{3}$. Plug this into the derivative: $\frac{dy}{dx} = -\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

32. **B** The Second Fundamental Theorem of Calculus tells us how to find the derivative of an integral. It says that $\frac{d}{dx}\int_c^u f(t)\,dt = f(u)\frac{du}{dx}$, where c is a constant and u is a function of x.

Here we can use the theorem to get

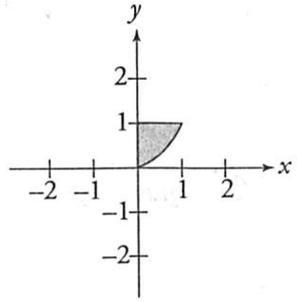
$$\frac{d}{dx}\int_0^{x^2} \sin^2 t\,dt = (\sin^2(x^2))(2x), \text{ or } 2x\sin^2(x^2)$$

33. **B** Here we have a function raised to another function. The best way to find the derivative is with logarithmic differentiation. First, take the log of both sides: $\ln y = \ln x^{\cos 4x}$. Next, use the log laws to rewrite the right-hand side: $\ln y = (\cos 4x)\ln x$. Now take the derivative of both sides: $\frac{1}{y}\frac{dy}{dx} = (\cos 4x)\left(\frac{1}{x}\right) - \ln x(4\sin 4x)$. Multiply both sides by y: $\frac{dy}{dx} = y\left[(\cos 4x)\left(\frac{1}{x}\right) - \ln x(4\sin 4x)\right]$. Remember that $y = x^{\cos 4x}$, so the derivative is $\frac{dy}{dx} = x^{\cos 4x}\left[(\cos 4x)\left(\frac{1}{x}\right) - \ln x(4\sin 4x)\right]$.

34. **C** We need to take the second derivative of f: $f'(x) = 1 - 2xe^{-x^2}$ and $f''(x) = -2x\left(-2xe^{-x^2}\right) - 2e^{-x^2} = 4x^2e^{-x^2} - 2e^{-x^2} = e^{-x^2}\left(4x^2 - 2\right)$. Now we set the second derivative equal to zero. Note that e^{-x^2} is always positive, so we get $4x^2 - 2 = 0$ and $x = \pm 0.707$.

35. **A** The width of each rectangle of the Riemann Sum will be $\frac{2-0}{4} = \frac{1}{2}$, so the sum is: $\frac{1}{2}\left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right]$. Evaluate this with a calculator and we get 26.357.

36. **D** First, make a quick sketch of the region.



We can find the volume by taking a vertical slice of the region. The formula for the volume of a solid of revolution around the x-axis, using a vertical slice bounded from above by the curve $f(x)$ and from below by $g(x)$, on the interval $[a, b]$, is

$$\pi\int_a^b \left[f(x)^2 - g(x)^2\right] dx$$

Here we get

$$\pi\int_0^1 [(1)^2 - (x^3)^2]\, dx$$

Now we have to evaluate the integral. First, expand the integrand to get

$$\pi\int_0^1 (1 - x^6)\, dx$$

Next integrate to get

$$\pi\left(x - \frac{x^7}{7}\right)\Bigg|_0^1 = \pi\left(1 - \frac{1}{7}\right) = \frac{6\pi}{7}$$

37. **B** We need an equation that relates the volume of a sphere to its radius, namely $V = \frac{4}{3}\pi r^3$. If we differentiate both sides with respect to t, we get $\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$. Next plug in $\frac{dV}{dt} = 20$ and $r = 4$: $20 = 4\pi(4)^2\frac{dr}{dt} = 64\pi\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{20}{64\pi} \approx 0.0995\frac{\text{in.}}{\text{s}}$.

38. **B** The Mean Value Theorem states that if a function is continuous on the interval $[a, b]$, then there exists a value c in the interval (a,b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$. Here, f is a polynomial, so it is continuous everywhere. First, let's find $\frac{f(b)-f(a)}{b-a}$. We get: $\frac{f(5)-f(1)}{5-1} = \frac{245-(-3)}{4} = 62$. Next, we need $f'(c)$. We get: $f'(x) = 6x^2$ so $f'(c) = 6c^2$. Now we set them equal to each other and solve: $6c^2 = 62$, so $c = \pm 3.215$. We don't use the negative value because it is not in the interval.

39. **D** Let's set $P = x^2y$. We want to maximize P, so we need to eliminate one of the variables. We are also given that $x + y = 100$, so we can solve this for y and substitute: $y = 100 - x$, so $P = x^2(100 - x) = 100x^2 - x^3$.

Now we can take the derivative.

$$\frac{dP}{dx} = 200x - 3x^2$$

Set the derivative equal to zero and solve for x.

$$200x - 3x^2 = 0$$
$$x(200 - 3x) = 0$$
$$x = 0 \text{ or } x = \frac{200}{3} \approx 66.667$$

Now we can use the second derivative to find the maximum: $\frac{d^2P}{dx^2} = 200 - 6x$. If we plug in $x = 66.667$, the second derivative is negative, so P is a maximum at $x = 66.667$. Solving for y, we get $y \approx 33.333$.

40. **D** First, we need to see if the velocity changes sign in the interval from $t = 0$ to $t = 2$. If you graph the velocity equation on your calculator, you will see that all of the y values are positive, so the object will always be moving to the right. Next, let's see how far the object travels in those 2 seconds. We can find this by evaluating $\int_0^2 t^2 \sin t\, dt$. If you plug this in your calculator, you get 2.469. (If you are unsure how to evaluate the integral on your calculator, check the Appendix for some tips on using a TI-84 calculator.) Because the object's initial position is 4, the position at time $t = 2$ will be 6.469.

41. **A** We can evaluate this integral using u-substitution. Let $u = \sin(\pi x)$ and $du = \pi\cos(\pi x)dx$, so $\frac{1}{\pi}du = \cos(\pi x)dx$. Substituting into the integrand, we get $\int \sin^4(\pi x)\cos(\pi x)dx = \frac{1}{\pi}\int u^4 du$. Evaluate: $\frac{1}{\pi}\int u^4 du = \frac{1}{\pi}\frac{u^5}{5} + C = \frac{u^5}{5\pi} + C$. Now we substitute back: $\frac{u^5}{5\pi} + C = \frac{\sin^5(\pi x)}{5\pi} + C$.

42. **D** If we want to find the amount that the balloon has inflated in the first 8 seconds, we need to evaluate $\int_0^8 300 - t\ln t\, dt$. Use your calculator to get 2349.458. (If you have a TI 84 series calculator, you can access the integration function by pressing MATH and then 9.) Add the original 100 in.^{3} to get 2449.458, which rounds to 2450 in.^{3}.

43. **A** We can find $\frac{dy}{dx}$ using implicit differentiation. We get: $2x+3x^2\frac{dy}{dx}+6xy+3y^2\frac{dy}{dx}=0$. Plug in the point (2, 1) and solve for $\frac{dy}{dx}$:

$$2(2)+3(2)^2\frac{dy}{dx}+6(2)(1)+3(1)^2\frac{dy}{dx}=0$$

$$4+12\frac{dy}{dx}+12+3\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{16}{15}$$

44. **B** First, we will need to find the y-coordinate that corresponds to $x = 1 : y = \tan 1 \approx 1.557$. Next, we need to find the derivative: $\frac{dy}{dx} = \tan x + x\sec^2 x$. We plug in $x = 1$ to get the slope of the tangent line: $\frac{dy}{dx} = \tan 1 + \sec^2 1 \approx 4.983$. Now we can plug this into the equation of a line: $y - 1.557 = 4.983(x - 1)$ or $y = 4.983x - 3.426$.

45. **C** In order to solve this for b, we need $f(x)$ to be continuous at $x = 2$. If we plug $x = 2$ into both pieces of this piecewise function, we get

$$f(x) = \begin{cases} 16a + 10; \; x \le 2 \\ 4b - 6; \; x > 2' \end{cases}$$

So, we need $16a + 10 = 4b - 6$. Now, if we take the derivative of both pieces of this function and plug in $x = 2$, we get

$$f'(x) = \begin{cases} 32a + 5; \; x \le 2 \\ 4b - 3; \; x > 2 \end{cases}, \text{ so we need } 32a + 5 = 4b - 3$$

Solving the simultaneous equations, we get $a = \frac{1}{2}$ and $b = 6$.

ANSWERS AND EXPLANATIONS TO SECTION II

1. The temperature on New Year's Day in Hinterland was given by $T(H) = -A - B\cos\left(\frac{\pi H}{12}\right)$, where T is the temperature in degrees Fahrenheit and H is the number of hours from midnight ($0 \le H < 24$).

(a) The initial temperature at midnight was –15° F, and at noon of New Year's Day was 5° F. Find A and B.

Simply plug in the temperature, –15, for T and the time, midnight ($H = 0$), for H into the equation. We get $-15 = -A - B\cos 0$, which simplifies to $-15 = -A - B$.

Now plug the temperature, 5, for T and the time, noon ($H = 12$), for H into the equation. We get $5 = -A - B\cos(\pi)$, which simplifies to $5 = -A + B$.

Now we can solve the pair of simultaneous equations for A and B, and we get $A = 5°\ F$ and $B = 10°\ F$.

(b) Find the average temperature for the first 10 hours.

In order to find the average value, we use the Mean Value Theorem for Integrals, which says that the average value of $f(x)$ on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Here, we have $\frac{1}{10-0}\int_0^{10}\left(-5 - 10\cos\left(\frac{\pi H}{12}\right)\right)dH$.

Evaluating the integral, we get

$$\frac{1}{10}\left[\left(-5H - \frac{120}{\pi}\sin\left(\frac{\pi H}{12}\right)\right)\right]_0^{10} = \frac{1}{10}\left[\left(-50 - \frac{120}{\pi}\sin\left(\frac{5\pi}{6}\right)\right)\right] = \frac{1}{10}\left[\left(-50 - \frac{60}{\pi}\right)\right] \approx -6.910°\text{F}$$

(c) Use the Trapezoid Rule with 4 equal subdivisions to estimate $\int_6^8 T(H)\,dH$.

The Trapezoid Rule enables us to approximate the area under a curve with a fair degree of accuracy. The rule says that the area between the x-axis and the curve $y = f(x)$ on the interval $[a, b]$, with n trapezoids, is

$$\frac{1}{2}\frac{b-a}{n}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n\right]$$

Using the rule here, with $n = 4$, $a = 6$, and $b = 8$, we get

$$\frac{1}{2}\cdot\frac{1}{2}\left[\left(-5-10\cos\frac{6\pi}{12}\right)+2\left(-5-10\cos\frac{13\pi}{24}\right)+2\left(-5-10\cos\frac{7\pi}{12}\right)+2\left(-5-10\cos\frac{15\pi}{24}\right)+\left(-5-10\cos\frac{8\pi}{12}\right)\right]$$

This is approximately –4.890°F.

(d) Find an expression for the rate that the temperature is changing with respect to H.

We simply take the derivative with respect to H.

$$\frac{dT}{dH} = -10\left(\frac{\pi}{12}\right)\left(-\sin\frac{\pi H}{12}\right) = \frac{5\pi}{6}\sin\frac{\pi H}{12}$$

2. Sea grass grows on a lake. The rate of growth of the grass is $\frac{dG}{dt} = kG$, where k is a constant.

(a) Find an expression for G, the amount of grass in the lake (in tons), in terms of t, the number of years, if the amount of grass is 100 tons initially and 120 tons after one year.

We solve this differential equation using separation of variables.

First, move the G to the left side and the dt to the right side, to get $\frac{dG}{G} = k\,dt$.

Now, integrate both sides.

$$\int\frac{dG}{G} = k\int dt$$

$$\ln G = kt + C$$

Next, solve for G by exponentiating both sides to the base e. We get $G = e^{kt+C}$.

Using the rules of exponents, we can rewrite this as $G = e^{kt}\,e^{C}$. Finally, because e^{C} is a constant, we can rewrite the equation as $G = Ce^{kt}$.

Now, we use the initial condition that $G = 100$ at time $t = 0$ to solve for C.

$$100 = Ce^{0} = C(1) = C$$

This gives us $G = 100e^{kt}$.

Next, we use the condition that $G = 120$ at time $t = 1$ to solve for k.

$$120 = 100e^{k}$$
$$1.2 = e^{k}$$
$$\ln 1.2 = k \approx 0.1823$$

This gives us $G = 100e^{0.1823t}$.

(b) In how many years will the amount of grass available be 300 tons?

All we need to do is set G equal to 300 and solve for t.

$$300 = 100e^{0.1823t}$$
$$3 = e^{0.1823t}$$
$$\ln 3 = 0.1823t$$
$$t \approx 6.026 \text{ years}$$

(c) If fish are now introduced into the lake and consume a consistent 80 tons/year of sea grass, how long will it take for the lake to be completely free of sea grass?

Now we have to account for the fish's consumption of the sea grass. So we have to evaluate the differential equation $\frac{dG}{dt} = kG - 80$.

First, separate the variables, to get

$$\frac{dG}{kG - 80} = dt$$

Now, integrate both sides.

$$\int \frac{dG}{kG - 80} = \int dt \text{ or } \int \frac{dG}{G - \frac{80}{k}} = k\int dt$$

$$\ln\left(G - \frac{80}{k}\right) = kt + C$$

Next, exponentiate both sides to the base e. We get

$$G - \frac{80}{k} = Ce^{kt}$$

Solving for G, we get

$$G = \left(G_0 - \frac{80}{k}\right)e^{kt} + \frac{80}{k}$$

Now, set $G = 0$. We get

$$0 = \left(G_0 - \frac{80}{k}\right)e^{kt} + \frac{80}{k}$$

Now, set $G_0 = 300$, and solve for e^{kt}.

$$e^{kt} = \frac{\frac{-80}{k}}{300 - \frac{80}{k}} = \frac{80}{80 - 300k}$$

Take the log of both sides.

$$kt = \ln\left(\frac{80}{80 - 300k}\right)$$

$$\text{and } t = \frac{1}{k}\ln\left(\frac{80}{80 - 300k}\right)$$

Now, we plug in the value for k that we got in part (a) above to get $t \approx 6.313$ years.

3. The functions f and g are twice-differentiable and have the following table of values:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	4	–2	–4
2	2	3	4	–2
3	5	2	–1	3
4	–1	–6	–8	0

(a) Let $h(x) = f(g(x))$. Find the equation of the tangent line to h at $x = 2$.

We can find the derivative of $h(x)$ using the Chain Rule: $h'(x) = f'(g(x))g'(x)$. At $x = 2$, we get $h'(2) = f'(g(2))g'(2) = f'(4)g'(2) = (-6)(4) = -24$.

And, at $x = 2$, $h(2) = f(g(2)) = f(4) = -1$.

Therefore, the equation of the tangent line is $y + 1 = -24(x - 2)$.

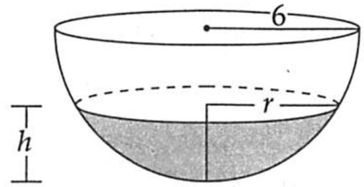
(b) Let $j(x) = f(x)g(x)$. Find $j'(3)$.

We can find the derivative of $j(x)$ using the Product Rule: $j'(x) = f(x)g'(x) + f'(x)g(x)$. At $x = 3$, we get: $j'(3) = f(3)g'(3) + f'(3)g(3) = (5)(3) + (2)(-1) = 13$.

(c) Evaluate $\int_1^4 3f''(x)\,dx$.

We simply evaluate $\int_1^4 3f''(x)\,dx = 3f'(x)\Big|_1^4 = 3f'(4) - 3f'(1) = 3(-6) - 3(4) = -30$.

4. Water is being poured into a hemispherical bowl of radius 6 inches at the rate of 4 in.³/sec.



(a) Given that the volume of the water in the spherical segment shown above is $V = \pi h^2\left(R - \frac{h}{3}\right)$, where R is the radius of the *sphere,* find the rate that the water level is rising when the water is 2 inches deep.

First, rewrite the equation as

$$V = \pi R h^2 - \frac{\pi}{3}h^3$$

Now, take the derivative of the equation with respect to t.

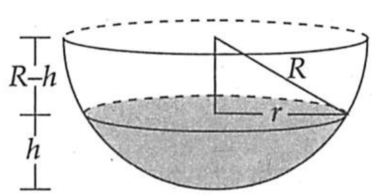
$$\frac{dV}{dt} = 2\pi R h\frac{dh}{dt} - \pi h^2\frac{dh}{dt}$$

If we plug in $\frac{dV}{dt} = 4$, $R = 6$, and $h = 2$, we get

$$4 = 20\pi\frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{4}{20\pi} = \frac{1}{5\pi}$$

(b) Find an expression for r, the radius of the *surface of the spherical segment* of water, in terms of h.

Notice that we can construct a right triangle using the radius of the sphere and the radius of the surface of the water.



Notice that the distance from the center of the sphere to the surface of the water is $R - h$. Now, we can use the Pythagorean Theorem to find r.

$$R^2 = (R - h)^2 + r^2$$

We can rearrange this to get

$$r = \sqrt{R^2 - (R - h)^2} = \sqrt{2Rh - h^2}$$

Because $R = 6$, we get

$$r = \sqrt{12h - h^2}$$

(c) How fast is the circular area of the surface of the spherical segment of water growing (in in.²/sec) when the water is 2 inches deep?

The area of the surface of the water is $A = \pi r^2$, where $r = \sqrt{12h - h^2}$. Thus, $A = \pi(12h - h^2)$.

Taking the derivative of the equation with respect to t, we get

$$\frac{dA}{dt} = \pi\left(12\frac{dh}{dt} - 2h\frac{dh}{dt}\right)$$

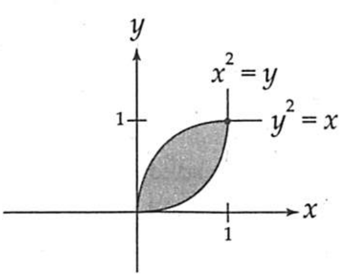
We found in part (a) above that

$$\frac{dh}{dt} = \frac{1}{5\pi}, \text{ so } \frac{dA}{dt} = \pi\left(\frac{12}{5\pi} - \frac{4}{5\pi}\right) = \frac{8}{5} \text{ in.}^2/\text{ sec}$$

5. Let R be the region in the first quadrant bounded by $y^2 = x$ and $x^2 = y$.

(a) Find the area of region R.

First, let's sketch the region.



In order to find the area, we "slice" the region vertically and add up all of the slices. Now, we use the formula for the area of the region between $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$.

$$\int_a^b \left[f(x) - g(x)\right] dx$$

We need to rewrite the equation $y^2 = x$ as $y = \sqrt{x}$ so that we can integrate with respect to x. Our integral for the area is

$$\int_0^1 \left(\sqrt{x} - x^2\right) dx$$

Evaluating the integral, we get

$$\left(\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

(b) Find the volume of the solid generated when R is revolved about the x-axis.

In order to find the volume of a region between $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$, when it is revolved around the x-axis, we use the following formula:

$$\pi\int_a^b \left[f(x)^2 - g(x)^2\right] dx$$

Here, our integral for the area is

$$\pi\int_0^1 \left(x - x^4\right) dx$$

Evaluating the integral, we get

$$\pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}$$

(c) The section of a certain solid cut by any plane perpendicular to the x-axis is a circle with the endpoints of its diameter lying on the parabolas $y^2 = x$ and $x^2 = y$. Find the volume of the solid.

Whenever we want to find the volume of a solid, formed by the region between $y = f(x)$ and $y = g(x)$, with a known cross-section, from $x = a$ to $x = b$, when it is revolved around the x-axis, we use the following formula:

$$\int_a^b A(x)\, dx$$

(Note: $A(x)$ is the area of the cross-section.) We find the area of the cross-section by using the vertical slice formed by $f(x) - g(x)$ and then plugging it into the appropriate area formula. In the case of a circle, $f(x) - g(x)$ gives us the length of the diameter and we use the following formula:

$$A(x) = \frac{\pi(diameter)^2}{4}$$

This gives us the integral,

$$\int_0^1 \frac{\pi}{4}\left(\sqrt{x} - x^2\right)^2 dx$$

Expand the integrand.

$$\int_0^1 \frac{\pi}{4}\left(\sqrt{x} - x^2\right)^2 dx = \frac{\pi}{4}\int_0^1\left(x - 2x^{\frac{5}{2}} + x^4\right) dx =$$

Evaluate the integral.

$$\frac{\pi}{4}\int_0^1\left(x - 2x^{\frac{5}{2}} + x^4\right) dx = \frac{\pi}{4}\left(\frac{x^2}{2} - \frac{4x^{\frac{7}{2}}}{7} + \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{\pi}{4}\left(\frac{1}{2} - \frac{4}{7} + \frac{1}{5}\right) = \frac{9\pi}{280}$$

6. For time $t \geq 0$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = 1 - 2\cos\left(\frac{t}{3}\right)$. The particle's position at time $t = 0$ is $x(0) = 8$.

(a) Is the particle speeding up or slowing down at time $t = 6$? Justify your answer.

An object is speeding up if its velocity and acceleration have the same sign at a given time, and it is slowing down if they have opposite signs at a given time. We can find the acceleration by taking the derivative of the velocity: $a(t) = \frac{2}{3}\sin\left(\frac{t}{3}\right)$. Now we can check the signs using a calculator: $v(6) = 1 - 2\cos(2) = 1.832$ and $a(6) = \frac{2}{3}\sin(2) = 0.606$. The acceleration and velocity have the same sign at time $t = 6$, so the particle is speeding up.

(b) When does the particle change direction in the interval $0 \leq t \leq 6$? Justify your answer.

The particle changes direction at times when the velocity is zero and changes sign. If we graph the velocity using a calculator, we can see that the velocity changes sign at $t = \pi$. To the left of $t = \pi$, the velocity is below the x-axis and to the right it is above, so the particle changes direction at the time $t = \pi$. (If you are unsure how to evaluate the integral on your calculator, check the Appendix for some tips on using a TI-84 calculator.)

(c) What is the particle's position at time $t = 6$?

We can find the change in the object's position in the first 6 seconds by evaluating the integral $\int_0^6 1 - 2\cos\left(\frac{t}{3}\right) dt$. Although we could do this by hand, we are allowed to use a calculator, which will make matters easier. We get $\int_0^6 1 - 2\cos\left(\frac{t}{3}\right) dt = 0.544$. Because the particle started at $x = 8$, the object's position at time $t = 6$ is 8.544. (If you are unsure how to evaluate the integral on your calculator, check the Appendix for some tips on using a TI-84 calculator.)

(d) What is the total distance traveled from time $t = 0$ to time $t = 6$?

If we want to find the total distance traveled in the first 6 seconds, we do something similar to what we did in part (c), but in the time between $t = 0$ and $t = \pi$, the particle is moving to the left (in part (b) we found that its velocity is negative), so we need to make that integral negative. Then we add it to the integral for when the particle is moving to the right between $t = \pi$ and $t = 6$. We get: $-\int_0^{\neq} 1 - 2\cos\left(\frac{t}{3}\right) dt + \int_{\neq}^6 1 - 2\cos\left(\frac{t}{3}\right)$. If we plug this into a calculator, we get: 2.055 + 2.599 = 4.654. (If you are unsure how to evaluate the integral on your calculator, check the Appendix for some tips on using a TI-84 calculator.)