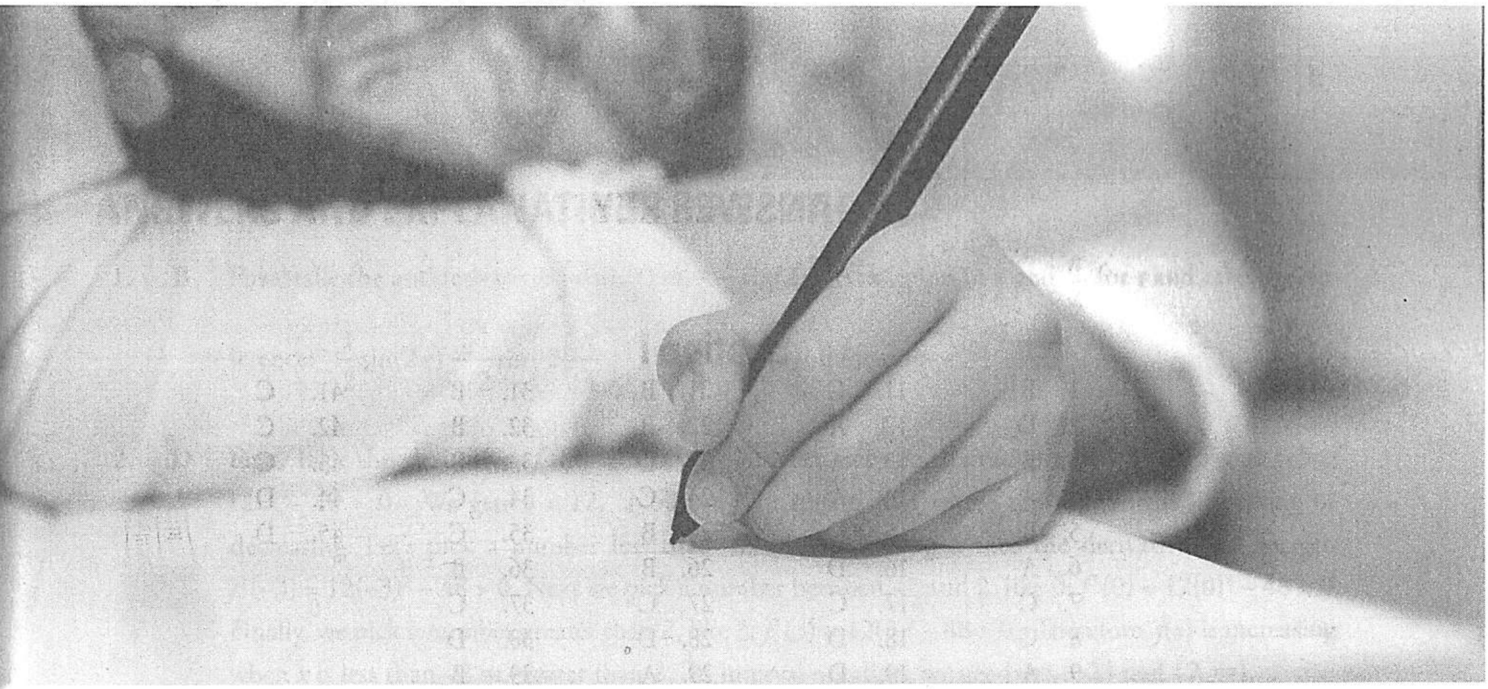


Practice Test 3: Answers and Explanations

ANSWER KEY

Section I

1. B	11. C	21. B	31. B	41. C
2. D	12. A	22. B	32. B	42. C
3. B	13. A	23. D	33. B	43. C
4. A	14. A	24. C	34. C	44. D
5. B	15. D	25. B	35. C	45. D
6. A	16. D	26. B	36. B	
7. C	17. C	27. C	37. C	
8. C	18. D	28. D	38. D	
9. A	19. D	29. A	39. B	
10. C	20. A	30. B	40. B	

ANSWERS AND EXPLANATIONS TO SECTION I

1. **B** First, take the antiderivative: $\int \cos(2t)\,dt = \frac{1}{2}\sin(2t)$. Next, plug in x and $\frac{\pi}{4}$ for t and take the difference: $\frac{1}{2}\sin(2x) - \frac{1}{2}\sin\left(2\left(\frac{\pi}{4}\right)\right)$. This can be simplified to $\frac{\sin(2x)-1}{2}$.

2. **D** First, take the derivative: $f'(x) = 12x^2 - 48$. Next, set the derivative equal to zero and solve: $12x^2 - 48 = 0$. We get: $x = \pm 2$. Now we sign test to find where the function is increasing or decreasing. Let's pick a number less than –2, like –3. Plug it into the derivative and we get: $f'(-3) = 12(-3)^2 - 48 > 0$. Next we pick a number between –2 and 2, like 0: $f'(0) = 12(0)^2 - 48 < 0$. Finally, we pick a number greater than 2, like 3: $f'(3) = 12(3)^2 - 48 > 0$. Therefore, $f(x)$ is increasing when x is less than –2 or greater than 2. In interval notation, we get $(-\infty, -2)$ and $(2, \infty)$.

3. **B** We need to use implicit differentiation to find $\frac{dy}{dx}$.

$$6x - 2\left(x\frac{dy}{dx} + y\right) + 3\frac{dy}{dx} = 0$$

$$6x - 2x\frac{dy}{dx} - 2y + 3\frac{dy}{dx} = 0$$

Now, if we wanted to solve for $\frac{dy}{dx}$ in terms of x and y, we would have to do some algebra to isolate $\frac{dy}{dx}$. But because we are asked to solve for $\frac{dy}{dx}$ at a specific value of x, we don't need to simplify. We need to find the y-coordinate that corresponds to the x-coordinate $x = 2$. We plug $x = 2$ into the original equation and solve for y.

$$3(2)^2 - 2(2)y + 3y = 12 - y = 1$$
$$y = 11$$

Finally, we plug $x = 2$ and $y = 11$ into the derivative, and we get

$$6(2) - 2(2)\frac{dy}{dx} - 2(11) + 3\frac{dy}{dx} = 0$$

$$12 - 4\frac{dy}{dx} - 22 + 3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -10$$

4. **A** Simplify the integrand by separating it into three fractions:

$$\int \frac{2x^3 + 4x^2 - 2x}{x^2}\,dx = \int \frac{2x^3}{x^2} + \frac{4x^2}{x^2} - \frac{2x}{x^2}\,dx$$

We can simplify this to $\int \frac{2x^3}{x^2} + \frac{4x^2}{x^2} - \frac{2x}{x^2}\,dx = \int 2x + 4 - \frac{2}{x}\,dx$. And integrate:

$$\int 2x + 4 - \frac{2}{x}\,dx = x^2 + 4x - 2\ln x + C$$

5. **B** We need to add the areas of the regions between the graph and the x-axis. Note that the area of the region between 0 and 5 has a positive value and the area of the region between 5 and 8 has a negative value. The area of the former region can be found by calculating the area of a trapezoid with bases of 2 and 5 and a height of 2. The area is $\frac{1}{2}(2 + 5)(2) = 7$. The area of the latter region can be found by calculating the area of a triangle with a base of 3 and a height of 2. The area is $\frac{1}{2}(3)(2) = 3$. Thus, the value of the integral is $7 - 3 = 4$.

6. **A** If we take the limit as x goes to 0, we get an indeterminate form $\frac{0}{0}$, so let's use L'Hôpital's Rule. We take the derivative of the numerator and the denominator to get $\lim_{x\to 0}\frac{x - \sin x}{x^2} = \lim_{x\to 0}\frac{1 - \cos x}{2x}$. When we take the limit, we again get an indeterminate form $\frac{0}{0}$, so let's use L'Hôpital's Rule a second time. We take the derivative of the numerator and the denominator and to $\lim_{x\to 0}\frac{1 - \cos x}{2x} = \lim_{x\to 0}\frac{\sin x}{2}$. Now, when we take the limit, we get $\lim_{x\to 0}\frac{\sin x}{2} = 0$.

7. **C** Use the Product Rule to find the derivative: $\frac{dy}{dx} = 2x(-\sin x) + 2\cos x = -2x\sin x + 2\cos x$.

8. **C** We can find the derivative with the Quotient Rule: $f'(x) = \frac{(2x-3)(3) - (3x-2)(2)}{(2x-3)^2} = \frac{-5}{(2x-3)^2}$.

9. **A** You should know that $\int \frac{dx}{x} = \ln|x| + C$. We take the antiderivative and we get $\int \left(\frac{4}{x-1}\right) dx = 4\ln|x - 1| + C$. Next, plug in $e + 1$ and 2 for x, and take the difference: $4\ln(e) - 4\ln(1)$. You should know that $\ln e = 1$ and $\ln 1 = 0$. Thus, we get $4\ln(e) - 4\ln(1) = 4$.

10. **C** We find the total distance traveled by finding the area of the region between the curve and the x-axis. Normally, we would have to integrate but here we can find the area of the region easily because it consists of geometric objects whose areas are simple to calculate. The area of the region between

$t = 0$ and $t = 4$ can be found by calculating the area of a triangle with a base of 4 and a height of 60. The area is $\frac{1}{2}(4)(60) = 120$. The area of the region between $t = 4$ and $t = 8$ can be found by calculating the area of a rectangle with a base of 4 and a height of 30. The area is $(4)(30) = 120$. The area of the region between $t = 8$ and $t = 16$ can be found by calculating the area of a trapezoid with bases of 4 and 8, and a height of 90 (or you could break it up into a rectangle and a triangle). The area is $\frac{1}{2}(4 + 8)(90) = 540$. Thus, the total distance traveled is $120 + 120 + 540 = 780$ kilometers.

11. **C** Use the Chain Rule: $f'(x) = \frac{1}{2}(\tan(5x))^{-\frac{1}{2}}(\sec^2(5x))(5)$. Next, plug in $x = \frac{\pi}{4}$:

$$f'\left(\frac{\pi}{4}\right) = \frac{1}{2}\left(\tan\left(\frac{5\pi}{4}\right)\right)^{-\frac{1}{2}}\left(\sec^2\left(\frac{5\pi}{4}\right)\right)(5) = \frac{1}{2}(1)^{-\frac{1}{2}}(2)(5) = 5$$

12. **A** If we want to find the equation of the tangent line, first we need to find the y-coordinate that corresponds to $x = \frac{\pi}{4}$. It is $y = \sin^2\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

Next, we need to find the derivative of the curve at $x = \frac{\pi}{4}$, using the Chain Rule.

We get $\frac{dy}{dx} = 2\sin x\cos x$. At $x = \frac{\pi}{4}$, $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 1$.

Now we have the slope of the tangent line and a point that it goes through. We can use the point-slope formula for the equation of a line, $(y - y_1) = m(x - x_1)$, and plug in what we have just found.

We get $\left(y - \frac{1}{2}\right) = (1)\left(x - \frac{\pi}{4}\right)$.

13. **A** Let's test whether f is continuous. First, $f(3) = 9$ exists. Next, $\lim_{x\to 3^-} f(x) = 9$ and $\lim_{x\to 3^+} f(x) = 9$, so $\lim_{x\to 3} f(x) = 9$. Finally, because $\lim_{x\to 3} f(x) = f(3) = 9$, $f(x)$ is continuous at $x = 3$.

14. **A** A graph is concave down where the second derivative is negative.

First, we find the first and second derivative.

$$\frac{dy}{dx} = 4x^3 + 24x^2 - 144x$$

$$\frac{d^2y}{dx^2} = 12x^2 + 48x - 144$$

Next, we want to determine on which intervals the second derivative of the function is positive and on which it is negative. We do this by finding where the second derivative is zero.

$$12x^2 + 48x - 144 = 0$$
$$x^2 + 4x - 12 = 0$$
$$(x + 6)(x - 2) = 0$$
$$x = -6 \text{ or } x = 2$$

We can test where the second derivative is positive and negative by picking a point in each of the three regions $-\infty < x < -6$, $-6 < x < 2$, and $2 < x < \infty$, plugging the point into the second derivative, and seeing what the sign of the answer is. You should find that the second derivative is negative on the interval $-6 < x < 2$.

15. **D** If we take the limit as x goes to ∞, we get an indeterminate form $\frac{\infty}{\infty}$, so let's use L'Hôpital's Rule. We take the derivative of the numerator and the denominator to get $\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x} = \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}} = \lim_{x\to\infty}\frac{x\ln 2}{x+1}$.

When we take the limit, we again get an indeterminate form, $\frac{\infty}{\infty}$, so let's use L'Hôpital's Rule a second time. We take the derivative of the numerator and the denominator to get $\lim_{x\to\infty}\frac{x\ln 2}{x+1} = \lim_{x\to\infty}\frac{\ln 2}{1} = \ln 2$.

16. **D** Here we want to examine the slopes of various pieces of the graph of $f(x)$. Notice that the graph has a positive slope from $x = -\infty$ to $x = -2$, where the slope is zero. Thus, we are looking for a graph of $f'(x)$ that is positive from $x = -\infty$ to $x = -2$ and zero at $x = -2$. Notice that the graph of $f(x)$ has a negative slope from $x = -2$ to $x = 2$, where the slope is zero. Thus, we are looking for a graph of $f'(x)$ that is negative from $x = -2$ to $x = 2$ and zero at $x = 2$. Finally, notice that the graph of $f(x)$ has a positive slope from $x = 2$ to $x = \infty$. Thus, we are looking for a graph of $f'(x)$ that is positive from $x = 2$ to $x = \infty$. The graph in (D) satisfies all of these requirements.

17. **C** Remember that $\frac{d}{dx}\ln(u(x)) = \frac{u'(x)}{u(x)}$.

We will need to use the Chain Rule to find the derivative.

$$f'(x) = \left(\frac{-\sin(3x)}{\cos(3x)}\right)(3) = -3\tan(3x)$$

18. **D** Use the Second Fundamental Theorem of Calculus: $\frac{d}{dx}\int_3^{4x} \cos^2 t\, dt = \cos^2(4x)(4)$.

19. **D** The particle is at rest where its velocity is zero. We can obtain the velocity from the derivative of the position function. We get: $v(t) = x'(t) = 6t^2 - 30t + 36$. Setting the velocity equal to zero we get: $6t^2 - 30t + 36 = 0$. Divide through by 6: $t^2 - 5t + 6 = 0$. $(t - 3)(t - 2) = 0$ So the particle is at rest at $t = 2$ and $t = 3$.

20. **A** We can use u-substitution to evaluate the integral. Let $u = \sin^2 x$ and $du = 2 \sin x \cos x\, dx$. Next, recall from trigonometry that $2 \sin x \cos x = \sin(2x)$. Now we can substitute into the integral $\int e^u\, du$, leaving out the limits of integration for the moment. Evaluate the integral to get $\int e^u\, du = e^u$. Now, we substitute back to get $e^{\sin^2 x}$. Finally, we evaluate at the limits of integration to get

$$e^{\sin^2 x}\Big|_0^{\frac{\pi}{2}} = e^{\sin^2 \frac{\pi}{2}} - e^{\sin^2 0} = e - 1$$

21. **B** In order to find the average value, we use the Mean Value Theorem for Integrals, which says that the average value of $f(x)$ on the interval $[a, b]$ is $\frac{1}{b - a}\int_a^b f(x)\,dx$.

Here we have $\frac{1}{\frac{\pi}{4} - \frac{\pi}{6}}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x\, dx$. Next, recall that $\frac{d}{dx}\tan x = \sec^2 x$.

We evaluate the integral.

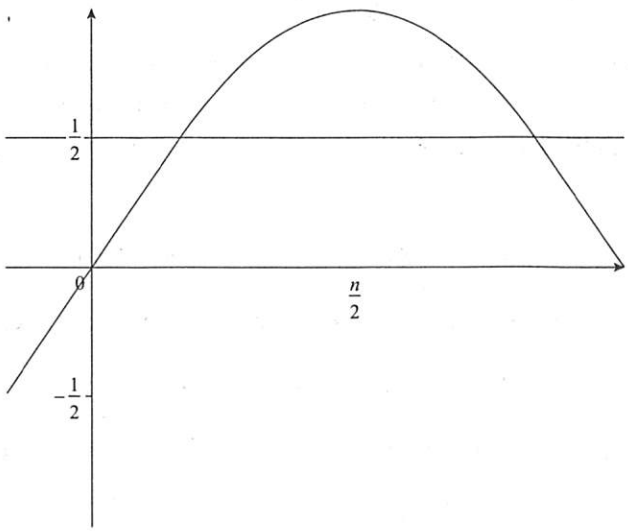
$$\frac{1}{\frac{\pi}{4} - \frac{\pi}{6}}(\tan x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{1}{\frac{\pi}{4} - \frac{\pi}{6}}\left[\tan\frac{\pi}{4} - \tan\frac{\pi}{6}\right] = \frac{1}{\frac{\pi}{4} - \frac{\pi}{6}}\left(1 - \frac{\sqrt{3}}{3}\right)$$

Get a common denominator for each of the two expressions.

$$\frac{1}{\frac{\pi}{4} - \frac{\pi}{6}}\left(1 - \frac{\sqrt{3}}{3}\right) = \frac{1}{\frac{6\pi}{24} - \frac{4\pi}{24}}\left(\frac{3}{3} - \frac{\sqrt{3}}{3}\right)$$

We can simplify this to $\frac{1}{\frac{2\pi}{24}}\left(\frac{3 - \sqrt{3}}{3}\right) = \frac{12}{\pi}\left(\frac{3 - \sqrt{3}}{3}\right) = \frac{12 - 4\sqrt{3}}{\pi}$.

22. **B** First, let's make a sketch of the region.



Note that the line intersects the curve at $x = \frac{\pi}{6}$, and that the line is above the curve for the whole region. So in order to find the area, we need to evaluate the integral

$$\int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \sin x\right) dx = \left(\frac{1}{2}x + \cos x\right)\Big|_0^{\frac{\pi}{6}} = \left(\frac{1}{2}\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\right) - \left(\frac{1}{2}(0) + \cos(0)\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

23. **D** A function is decreasing on an interval where the derivative is negative. The derivative is $f'(x) = 4x^3 + 12x^2$. We want to determine on which intervals the derivative of the function is positive and on which it is negative. We do this by finding where the derivative is zero.

$$4x^3 + 12x^2 = 0$$

$$4x^2(x + 3) = 0$$

$$x = -3 \text{ or } x = 0$$

We can test where the derivative is positive and negative by picking a point in each of the three regions $-\infty < x < -3$, $-3 < x < 0$, and $0 < x < \infty$, plugging the point into the derivative, and seeing what the sign of the answer is. Because x^2 is never negative, you should find that the derivative is negative on the interval $-\infty < x < -3$.

24. **C** We will need to use the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ to find the limit.

First, rewrite the limit as $\lim_{x \to 0} \frac{\frac{\sin(3x)}{\cos(3x)} + 3x}{\sin(5x)} =$

Next, break the expression into two rational expressions.

$$\lim_{x \to 0} \frac{\sin(3x)}{\sin(5x)\cos(3x)} + \frac{3x}{\sin(5x)} =$$

This can be broken up further into

$$\lim_{x \to 0} \frac{\sin(3x)}{\sin(5x)} \frac{1}{\cos(3x)} + \frac{3x}{\sin(5x)} =$$

We will evaluate the limit of each separately.

<u>First expression</u>

Divide the top and bottom by x: $\lim_{x \to 0} \frac{\frac{\sin(3x)}{x}}{\frac{\sin(5x)}{x}}$.

Then, multiply the top and bottom of the upper expression by 3, and the top and bottom of the lower expression by 5: $\lim_{x \to 0} \frac{\frac{3\sin(3x)}{3x}}{\frac{5\sin(5x)}{5x}}$.

Now, if we take the limit, we get $\lim_{x \to 0} \frac{\frac{3\sin(3x)}{3x}}{\frac{5\sin(5x)}{5x}} = \frac{3(1)}{5(1)} = \frac{3}{5}$.

<u>Second expression</u>

This limit is straightforward: $\lim_{x \to 0} \frac{1}{\cos(3x)} = \frac{1}{\cos(0)} = 1$.

<u>Third expression</u>

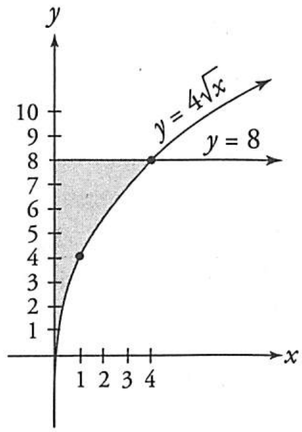
First, pull the constant, 3, out of the limit: $\lim_{x \to 0} \frac{3x}{\sin(5x)} = 3\lim_{x \to 0} \frac{x}{\sin(5x)}$.

Now, if we multiply the top and bottom of the expression by 5, we get $3\lim_{x \to 0} \frac{5x}{5\sin(5x)}$.

Now, if we take the limit, we get $3\lim_{x \to 0} \frac{5x}{5\sin(5x)} = 3\left(\frac{1}{5}\right) = \frac{3}{5}$.

Combine the three numbers to get $\frac{3}{5}(1) + \frac{3}{5} = \frac{6}{5}$.

25. **B** First, we graph the curves.



We can find the volume by taking a vertical slice of the region. The formula for the volume of a solid of revolution around the x-axis, using a vertical slice bounded from above by the curve $f(x)$ and from below by $g(x)$, on the interval $[a, b]$, is

$$\pi\int_a^b\left[f(x)^2 - g(x)^2\right]dx$$

The upper curve is $y = 8$, and the lower curve is $y = 4\sqrt{x}$.

Next, we need to find the point(s) of intersection of the two curves, which we do by setting them equal to each other and solving for x.

$$8 = 4\sqrt{x}$$

$$2 = \sqrt{x}$$

$$x = 4$$

Thus, the limits of integration are $x = 0$ and $x = 4$.

Now, we evaluate the integral.

$$\pi\int_0^4\left[(8)^2 - \left[4\sqrt{x}\right]^2\right]dx = \pi\int_0^4\left(64 - 16x\right)dx = \pi\left(64x - 8x^2\right)\Big|_0^4 = 128\pi$$

26. **B** Velocity is the first derivative of position with respect to time.

The first derivative is

$$v(t) = 6t^2 - 24t + 16$$

If we want to find the maximum velocity, we take the derivative of velocity (which is acceleration) and find where the derivative is zero.

$$v'(t) = 12t - 24$$

Next, we set the derivative equal to zero and solve for t, in order to find the critical value.

$$12t - 24 = 0$$

$$t = 2$$

Note that the second derivative of velocity is 12, which is positive. Remember the second derivative test: If the sign of the second derivative at a critical value is positive, then the curve has a local minimum there. If the sign of the second derivative is negative, then the curve has a local maximum there.

Thus, the velocity is a *minimum* at $t = 2$. In order to find where it has an absolute *maximum,* we plug the endpoints of the interval into the original equation for velocity, and the larger value will be the answer. At $t = 0$, the velocity is 16. At $t = 5$, the velocity is 46.

27. **C** First, we need to find $\frac{f(6)-f(2)}{(6-2)} = \frac{48-8}{4} = 10$. Next, we take the derivative at $x = c$. We get $f'(x) = 2x + 2$ and $f'(c) = 2c + 2$. Setting them equal to each other, we get $2c + 2 = 10$ and $c = 4$.

28. **D** Remember that $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, so here we get $\frac{d}{dx}\sin^{-1}(4x) = \frac{1}{\sqrt{1-(4x)^2}}(4) = \frac{4}{\sqrt{1-16x^2}}$.

29. **A** The Second Fundamental Theorem of Calculus tells us how to find the derivative of an integral: $\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt$. Here we can use the theorem to get $\frac{d}{dx}\int_{2x}^{5x} \cos t \, dt = 5\cos 5x - 2\cos 2x$.

30. **B** The function $g(x) = \int_0^x f(t)\,dt$ is called an accumulation function and stands for the area between the curve and the x-axis to the point x. At $x = 0$, the area is 0, so $g(0) = 0$. From $x = 0$ to $x = 2$, the area grows, so $g(x)$ has a positive slope. Then, from $x = 2$ to $x = 4$, the area shrinks (because we subtract the area of the region under the x-axis from the area of the region above it), so $g(x)$ has a negative slope. Finally, from $x = 4$ to $x = 6$, the area again grows, so $g(x)$ has a positive slope. The curve that best represents this is shown in (B).

31. **B** The slope of the tangent line is the derivative of the function. We get $f'(x) = 3e^{3x}$. Now we set the derivative equal to 2 and solve for x.

$$3e^{3x} = 2$$

$$e^{3x} = \frac{2}{3}$$

$$3x = \ln\frac{2}{3}$$

$$x = \frac{1}{3}\ln\frac{2}{3} \approx -0.135$$

Remember to round all answers to three decimal places on the AP Exam.

32. **B** First, let's find the derivative: $\frac{dy}{dx} = 3x^2 + 24x + 15$.

Next, set the derivative equal to zero and solve for x.

$$3x^2 + 24x + 15 = 0$$
$$x^2 + 8x + 5 = 0$$

Using the quadratic formula (or a calculator), we get

$$x = \frac{-8 \pm \sqrt{64 - 20}}{2} \approx -0.683, -7.317$$

Let's use the second derivative test to determine which is the maximum. We take the second derivative and then plug in the critical values that we found when we set the first derivative equal to zero. If the sign of the second derivative at a critical value is positive, then the curve has a local minimum there. If the sign of the second derivative is negative, then the curve has a local maximum there.

The second derivative is $\frac{d^2y}{dx^2} = 6x + 24$. The second derivative is negative at $x = -7.317$, so the curve has a local maximum there.

33. **B** The formula for the perimeter of a square is $P = 4s$, where s is the length of a side of the square.

If we differentiate this with respect to t, we get $\frac{dP}{dt} = 4\frac{ds}{dt}$. We plug in $\frac{ds}{dt} = 0.4$ to get $\frac{dP}{dt} = 4(0.4) = 1.6$.

The formula for the area of a square is $A = s^2$. If we solve the perimeter equation for s in terms of P and substitute it into the area equation, we get

$$s = \frac{P}{4}, \text{ so } A = \left(\frac{P}{4}\right)^2 = \frac{P^2}{16}$$

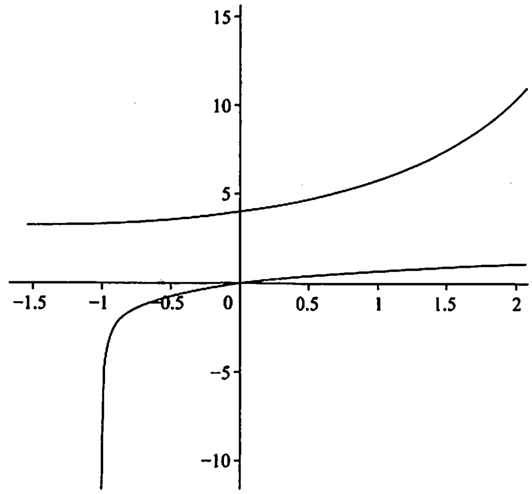
If we differentiate this with respect to t, we get $\frac{dA}{dt} = \frac{P}{8}\frac{dP}{dt}$.

Now we plug in $\frac{dP}{dt} = 1.6$ to get $\frac{dA}{dt} = \frac{P}{8}(1.6) = 0.2P$.

34. **C** The slope of the tangent line is the derivative of the function. Recall that $\frac{d}{dx}a^x = a^x \ln a$. Here we get $f'(x) = 3^x \ln 3$. Now we set the derivative equal to 1 and solve for x. Using the calculator, we get $3^x \ln 3 = 1$, so $x \approx -0.086$.

35. **C** First, use the Chain Rule to find $h'(x) = f'(g(x))g'(x)$. Next, evaluate the derivative at $x = 2$. We get $h'(2) = f'(g(2))g'(2)$. Use the table to find $g(2) = 1$ and $g'(2) = 6$. Next, because $g(2) = 1$, we need to find $f'(1) = 4$ (from the table). Therefore, $h'(2) = f'(g(2))g'(2) = f'(1)g'(2) = (4)(6) = 24$.

36. **B** First, graph the equations to see which one is on top and which is on the bottom.



Note that $f(x) = e^x + 3$ is always above $g(x) = \ln(x + 1)$, so in order to find the area, we simply have to integrate $\int_0^2 \left(e^x + 3\right) - \ln\left(x+1\right)dx$. If you plug this in your calculator, you get 11.093. (If you are unsure how to evaluate the integral on your calculator, check the Appendix for some tips on using a TI-84 calculator.)

37. **C** The Trapezoid Rule enables us to approximate the area under a curve with a fair degree of accuracy. The rule says that the area between the x-axis and the curve $y = f(x)$, on the interval $[a, b]$, with n trapezoids, is

$$\frac{1}{2}\frac{b-a}{n}[y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n]$$

Using the rule here, with $n = 4$, $a = 0$, and $b = 3$, we get

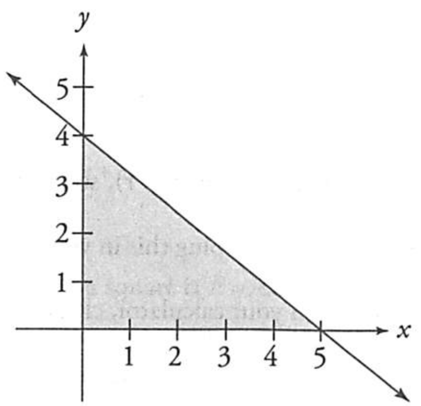
$$\frac{1}{2}\left(\frac{3}{4}\right)\left[e^0 + 2e^{\frac{3}{4}} + 2e^{\frac{6}{4}} + 2e^{\frac{9}{4}} + e^3\right] \approx 19.972$$

38. **D** The function will be concave up where the second derivative is positive. We need to differentiate f twice: $f'(x) = 4x^3 - 6x$ $f''(x) = 12x^2 - 6$. Now we set the second derivative equal to zero and solve. We get $x = \pm\sqrt{\frac{1}{2}}$. Now we just have to sign test the different intervals. Pick a number less than $-\sqrt{\frac{1}{2}}$, say −1, and plug it into the second derivative: $f''(-1) = 12(-1)^2 - 6 > 0$, so f is concave up for x

less than $-\sqrt{\frac{1}{2}}$. Next try a number between $-\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{2}}$, say 0: $f''(0)=12(0)^2-6<0$, so f is concave down on that interval. Finally, pick a number greater than $\sqrt{\frac{1}{2}}$, say 1: $f''(1)=12(1)^2-6>0$, so f is concave up for x greater than $\sqrt{\frac{1}{2}}$. Therefore, f is concave up on $\left(-\infty,-\sqrt{\frac{1}{2}}\right)$ and $\left(\sqrt{\frac{1}{2}},\infty\right)$.

39. **B** If we plug $x=\frac{\pi}{2}$ into the limit, we get $\frac{1-\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}=\frac{0}{0}$, which is an indeterminate form. This means that we can use L'Hôpital's Rule. Take the derivative of the numerator and denominator: $\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x}$. Now if we evaluate the limit, we get $\frac{-\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}}=0$.

40. **B** First, sketch the region.



The rule for finding the volume of a solid with known cross-sections is $V=\int_a^b A(x)\,dx$, where A is the formula for the area of the cross-section. So x represents the diameter of a semi-circular cross-section.

The area of a semi-circle in terms of its diameter is $A=\pi\frac{d^2}{8}$. We find the length of the diameter by solving the equation $4x + 5y = 20$ for y: $y=\frac{20-4x}{5}$. Next, we need to find where the graph intersects the x-axis. You should get $x = 5$. Thus, we find the volume by evaluating the integral.

$$\int_0^5 \pi\frac{\left(\frac{20-4x}{5}\right)^2}{8}\,dx$$

This integral can be simplified to

$$\frac{\pi}{200}\int_0^5 (20-4x)^2\,dx = \frac{\pi}{200}\int_0^5 \left(400-160x+16x^2\right)dx$$

You can evaluate the integral by hand or with a calculator. You should get

$$\frac{\pi}{200}\int_0^5 \left(400-160x+16x^2\right)dx = \frac{10\pi}{3}$$

41. **C** If we want to find the equation of the tangent line, first we need to find the x-coordinate that corresponds to $y = 3$. If you use your calculator to solve $x^3 + x^2 = 3$, you should get $x = 1.1746$.

Next, we need to find the derivative of the curve at $x = 1.1746$. We get

$$\frac{dy}{dx} = 3x^2 + 2x. \text{ At } x = 1.1746, \left.\frac{dy}{dx}\right|_{x=1.1746} = 3(1.1746)^2 + 2(1.1746) = 6.488$$

(It is rounded to three decimal places.)

Now we have the slope of the tangent line and a point that it goes through. We can use the point-slope formula for the equation of a line, $(y - y_1) = m(x - x_1)$, and plug in what we have just found. We get $(y - 3) = (6.488)(x - 1.1746)$. This simplifies to $y = 6.488x - 4.620$.

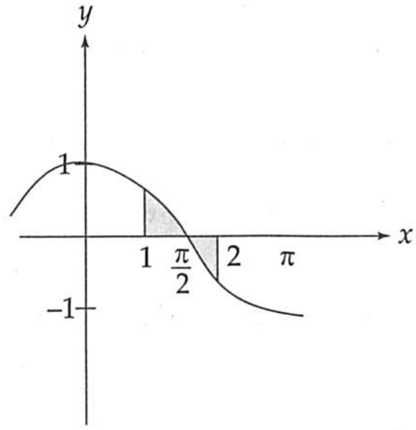
42. **C** Set the derivative equal to zero and solve for x. Using your calculator, you should get $\ln x - x + 2 = 0$.

$x = 3.146$ or $x = 0.159$ (rounded to three decimal places)

Let's use the second derivative test to determine which is the minimum. We take the second derivative and then plug in the critical values that we found when we set the first derivative equal to zero. If the sign of the second derivative at a critical value is positive, then the curve has a local minimum there. If the sign of the second derivative is negative, then the curve has a local maximum there.

The second derivative is $f''(x) = \frac{1}{x} - 1$. The second derivative is positive at $x = 0.159$, so the curve has a local minimum there.

43. **C** First, we should graph the curve.



Note that the curve is above the x-axis from $x = 1$ to $x = \frac{\pi}{2}$ and below the x-axis from $x = \frac{\pi}{2}$ to $x = 2$. Thus, we need to evaluate two integrals to find the area.

$$\int_1^{\frac{\pi}{2}} \cos x \, dx + \int_{\frac{\pi}{2}}^{2} (-\cos x) \, dx$$

We will need a calculator to evaluate these integrals.

$$\int_1^{\frac{\pi}{2}} \cos x \, dx + \int_{\frac{\pi}{2}}^{2} (-\cos x) \, dx \approx 0.249$$

44. **D** First, we find $\int \cot x \, dx$ by rewriting the integral as $\int \frac{\cos x}{\sin x} dx$. Then, we use u-substitution. Let $u = \sin x$ and $du = \cos x$. Substituting, we can get $\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln|u| + C$. Then substituting back, we get $\ln(\sin x) + C$. (We can get rid of the absolute value bars because sine is always positive on the interval.)

Next, we use $f\left(\frac{\pi}{6}\right) = 1$ to solve for C. We get $1 = \ln\left(\sin \frac{\pi}{6}\right) + C$.

$$1 = \ln\left(\frac{1}{2}\right) + C$$

$$1 = \ln\left(\frac{1}{2}\right) + C = 1.693147$$

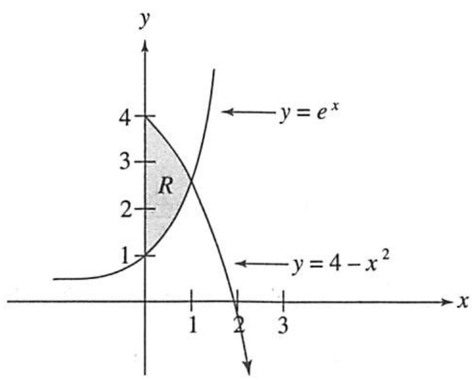
Thus, $f(x) = \ln(\sin x) + 1.693147$.

At $x = 1$, we get $f(1) = \ln(\sin 1) + 1.693147 = 1.521$ (rounded to three decimal places).

45. **D** We can solve this differential equation by first separating the variables: $\frac{dy}{dx} = y(2x-1)$ $\frac{dy}{y} = (2x-1)dx$. Next, we integrate both sides: $\int \frac{dy}{y} = \int (2x-1)dx$ $\ln|y| = x^2 - x + C$. Now we exponentiate both sides: $y = e^{x^2-x+C} = Ce^{x^2-x}$. Finally, plug in the initial condition to solve for *C:* $4 = Ce^0 = C$, so we get $y = 4e^{x^2-x}$

ANSWERS AND EXPLANATIONS TO SECTION II

1.



Let *R* be the region in the first quadrant shown in the figure above.

(a) Find the area of *R*.

In order to find the area, we "slice" the region vertically and add up all of the slices. We use the formula for the area of the region between $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$,

$$\int_a^b [f(x) - g(x)]\, dx$$

We have

$$f(x) = 4 - x^2 \text{ and } g(x) = e^x$$

Next, we need to find the point of intersection in the first quadrant. Use your calculator to find that the point of intersection is $x = 1.058$ (rounded to three decimal places). Plugging into the formula, we get

$$\int_0^{1.058} \left[(4 - x^2) - e^x\right] dx$$

Evaluating the integral, we get $\int_0^{1.058} \left[(4 - x^2) - e^x\right] dx = \left(4x - \frac{x^3}{3} - e^x\right)\Big|_0^{1.058} = 1.957.$

(b) Find the volume of the solid generated when R is revolved about the x-axis.

In order to find the volume of a region between $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$, when it is revolved about the x-axis, we use the following formula:

$$\pi \int_a^b \left[f(x)^2 - g(x)^2\right] dx$$

Here our integral is $\pi \int_0^{1.058} \left[\left(4 - x^2\right)^2 - \left(e^x\right)^2\right] dx.$

Evaluating the integral, we get

$$\pi \int_0^{1.058} \left[\left(4 - x^2\right)\right]^2 - \left[e^x\right]^2 dx = \pi \int_0^{1.058} \left(16 - 8x^2 + x^4 - e^{2x}\right) dx = \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} - \frac{e^{2x}}{2}\right]_0^{1.058} = 32.629$$

(c) Find the volume of the solid generated when R is revolved about the line $x = -1$.

In order to find the volume of this region, if we want to use vertical slices, we will use the method of cylindrical shells. Also, because we are revolving about the line $x = -1$, we will need to add 1 to the radius of the cylindrical shell. We will use the formula

$$2\pi \int_a^b (x + 1)\left[f(x) - g(x)\right] dx$$

We get

$$2\pi \int_0^{1.058} (x + 1)\left[(4 - x^2) - e^x\right] dx$$

We suggest that you use your calculator to evaluate the integral.

$$2\pi \int_0^{1.058} (x + 1)\left[(4 - x^2) - e^x\right] dx = 2\pi \int_0^{1.058} \left[4x - x^3 - xe^x + 4 - x^2 - e^x\right] dx = 17.059$$

2. A body is coasting to a stop and the only force acting on it is a resistance proportional to its speed, according to the equation $\frac{ds}{dt} = v_f = v_0\, e^{-\left(\frac{k}{m}\right)t}$; $s(0) = 0$, where v_0 is the body's initial velocity (in m/s), v_f is its final velocity, m is its mass, k is a constant, and t is time.

(a) If a body, with mass $m = 50$ kg and $k = 1.5$ kg/s, initially has a velocity of 30 m/s, how long, to the nearest second, will it take to slow to 1 m/s?

We simply plug into the formula and solve for t.

$$v_f = v_0\, e^{-\left(\frac{k}{m}\right)t}$$

$$1 = 30\, e^{-\left(\frac{1.5}{50}\right)t}$$

Divide both sides by 30: $\frac{1}{30} = e^{-\left(\frac{1.5}{50}\right)t}$.

Take the log of both sides: $\ln\frac{1}{30} = -\left(\frac{1.5}{50}\right)t$.

Multiply both sides by $-\frac{50}{1.5}$: $-\frac{50}{1.5}\ln\frac{1}{30} = t \approx 113$ seconds.

(b) How far, to the nearest 10 meters, will the body coast during the time it takes to slow from 30 m/s to 1 m/s?

Now we need to solve the differential equation $\frac{ds}{dt} = v_0\, e^{-\left(\frac{k}{m}\right)t}$, which we can do with separation of variables.

First, multiply both sides by dt: $ds = v_0\, e^{-\left(\frac{k}{m}\right)t}\, dt$.

Integrate both sides: $\int ds = \int v_0\, e^{-\left(\frac{k}{m}\right)t}\, dt$.

Evaluate the integrals: $s = -\frac{mv_0}{k}\, e^{-\left(\frac{k}{m}\right)t} + C$.

Now, plug in the initial conditions to solve for C: $0 = -\frac{(50)(30)}{1.5}\, e^{-\left(\frac{1.5}{50}\right)(0)} + C$.

$$C = \frac{(30)(50)}{1.5} = 1{,}000$$

Therefore, $s = -\frac{mv_0}{k}e^{-\left(\frac{k}{m}\right)t} + 1{,}000$. Now, we plug in the time $t = 113$ that we found in part (a) as well as the initial conditions to solve for s, which yields the following:

$$s = -\frac{(50)(30)}{1.5}e^{-\left(\frac{1.5}{50}\right)113} + 1{,}000 \approx 970 \text{ meters}$$

(c) If the body coasts from 30 m/s to a stop, how far will it coast?

Here, because the braking force is an exponential function, the object will coast to a stop after an infinite amount of time. In other words, we need to find

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty}\left[1{,}000 - 1{,}000e^{-\left(\frac{k}{m}\right)t}\right] = 1{,}000 \text{ meters.}$$

3. An object moves with velocity $v(t) = 9t^2 + 18t - 7$ for $t \geq 0$ from an initial position of $s(0) = 3$.

(a) Write an equation for the position of the particle.

The position function of the particle can be determined by integrating the velocity with respect to time, thus $s(t) = \int v(t)dt$. For this problem, $s(t) = \int (9t^2 + 18t - 7)dt = 3t^3 + 9t^2 - 7t + C$. Because we are given the initial position, $s(0) = 3$, plug that in to solve for C. Thus, $C = 3$ and the equation for the position of the particle is $s(t) = 3t^3 + 9t^2 - 7t + 3$.

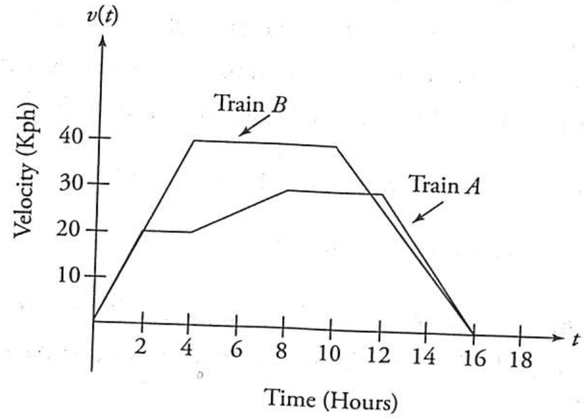
(b) When is the particle changing direction?

The particle changes direction when the velocity is zero, but the acceleration is not. In order to determine when those times are, set the velocity equal to zero and solve for t: $v(t) = 9t^2 + 18t - 7 = 0$ when $t = \frac{1}{3}$ and $t = -\frac{7}{3}$. Because the time range in question is $t \geq 0$, we can ignore $t = -\frac{7}{3}$. Then, take the derivative of the velocity function to find the acceleration function, as $\frac{d}{dt}(v(t)) = a(t)$. For the given $v(t)$, $a(t) = 18t + 18$. Check that the acceleration at time $t = \frac{1}{3}$ is not zero by plugging into the acceleration function: $a(t) = 24$. Therefore, the particle is changing direction at $t = \frac{1}{3}$ because $v(t) = 0$ and $a(t) \neq 0$.

(c) What is the total distance covered from $t = 2$ to $t = 5$?

The distance covered is found by using the position function found in part (a). Determine the position at $t = 2$ and subtract it from the position at $t = 5$. From part (b), we know that the object does not change direction over this time interval, so we do not need to find the time piecewise. Thus, $s(5) - s(2) = 568 - 49 = 519$.

4.



Three trains, A, B, and C, each travel on a straight track for $0 \leq t \leq 16$ hours. The graphs above, which consist of line segments, show the velocities, in kilometers per hour, of trains A and B. The velocity of C is given by $v(t) = 8t - 0.25t^2$.

(Indicate units of measure for all answers.)

(a) Find the velocities of A and C at time $t = 6$ hours.

We can find the velocity of train A at time $t = 6$ simply by reading the graph. We get $v_A(6) = 25$ kph. We find the velocity of train C at time $t = 6$ by plugging $t = 6$ into the formula. We get $v_C(6) = 8(6) - 0.25(6^2) = 39$ kilometers per hour.

(b) Find the accelerations of B and C at time $t = 6$ hours.

Acceleration is the derivative of velocity with respect to time. For train B, we look at the *slope* of the graph at time $t = 6$. We get $a_B(6) = 0$ km/hr². For train C, we take the derivative of v. We get $a(t) = 8 - 0.5t$. At time $t = 6$, we get $a_C(6) = 5$ km/hr².

(c) Find the positive difference between the total distance that A traveled and the total distance that B traveled in 16 hours.

In order to find the total distance that train A traveled in 16 hours, we need to find the area under the graph. We can find this area by adding up the areas of the different geometric objects that are under the graph. From time $t = 0$ to $t = 2$, we need to find the area of a triangle with a base of 2 and a height of 20. The area is 20. Next, from time $t = 2$ to $t = 4$, we need to find the area of a rectangle with a base of 2 and a height of 20. The area is 40. Next, from time $t = 4$ to $t = 8$, we need to find the area of a trapezoid with bases of 20 and 30 and a height of 4. The area is 100. Next, from time $t = 8$ to $t = 12$, we need to find the area of a rectangle with a base of 4 and a height of 30. The area is 120. Finally, from time $t = 12$ to $t = 16$, we need to find the area of a triangle with a base of 4 and a height of 30. The area is 60. Thus, the total distance that train A traveled is 340 km.

Let's repeat the process for train B. From time $t = 0$ to $t = 4$, we need to find the area of a triangle with a base of 4 and a height of 40. The area is 80. Next, from time $t = 4$ to $t = 10$, we need to find the area of a rectangle with a base of 6 and a height of 40. The area is 240. Finally, from time $t = 10$ to $t = 16$, we need to find the area of a triangle with a base of 6 and a height of 40. The area is 120. Thus, the total distance that train B traveled is 440 km.

Therefore, the positive difference between their distances is 100 km.

(d) Find the total distance that C traveled in 16 hours.

First, note that the graph of train C's velocity, $v(t) = 8t - 0.25t^2$, is above the x-axis on the entire interval. Therefore, in order to find the total distance traveled, we integrate $v(t)$ over the interval. We get

$$\int_0^{16} (8t - 0.25t^2)\, dt$$

Evaluate the integral: $\int_0^{16} (8t - 0.25t^2)\, dt = \left(4t^2 - \frac{t^3}{12}\right)\Big|_0^{16} = \frac{2,048}{3}$ km.

5. Consider the curve $y^2 + 3xy = 4$.

(a) Find an equation of the tangent line to the curve at the point (1, 1).

First, we need to use Implicit Differentiation to find $\frac{dy}{dx}$. We get: $2y\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) = 0$. Let's isolate $\frac{dy}{dx}$ because we will need it in part (c). Otherwise, we would plug in (1, 1) here to solve for the slope.

$$2y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

$$\frac{dy}{dx}(2y + 3x) = -3y$$

$$\frac{dy}{dx} = \frac{-3y}{2y + 3x}$$

Now plug in (1, 1) to solve for the slope: $\frac{dy}{dx} = \frac{-3}{2+3} = -\frac{3}{5}$.

Therefore, the equation of the tangent line is $y - 1 = -\frac{3}{5}(x - 1)$.

(b) Find all x-coordinates where the slope of the tangent line to the curve is undefined.

The tangent line will be undefined where the denominator is zero. That is,

$$2y + 3x = 0$$

$$2y = -3x$$

$$y = -\frac{3x}{2}$$

Plug $y = -\frac{3x}{2}$ into the original equation: $\left(-\frac{3x}{2}\right)^2 - 3x\left(-\frac{3x}{2}\right) = 4$. And solve:

$$\frac{9x^2}{4} + \frac{9x^2}{2} = 4$$

$$\frac{27x^2}{4} = 4$$

$$x^2 = \frac{16}{27}$$

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point (1, 1).

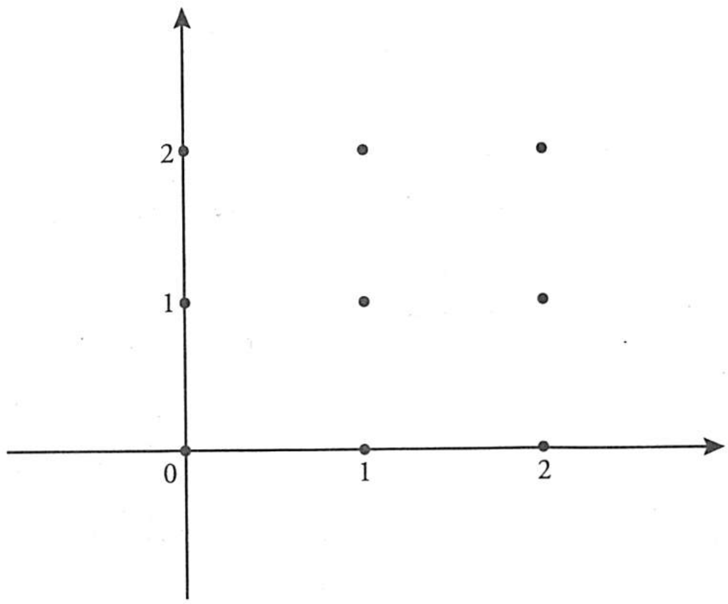
We need to take the derivative of $\frac{dy}{dx}$ that we found in part (a). Using the Quotient Rule and Implicit Differentiation, we get: $\frac{d^2y}{dx^2} = \frac{(2y+3x)\left(-3\frac{dy}{dx}\right) - 3y\left(2\frac{dy}{dx}+3\right)}{(2y+3x)^2}$.

Note that at (1, 1), $\frac{dy}{dx} = -\frac{3}{5}$. Therefore, at (1, 1),

$$\frac{d^2y}{dx^2} = \frac{(2+3)\left(-3\left(\frac{-3}{5}\right)\right) - 3\left(2\left(\frac{-3}{5}\right)+3\right)}{(2+3)^2} = \frac{(5)\left(\frac{9}{5}\right) - (3)\left(\frac{9}{5}\right)}{25} = \frac{8}{125}$$

6. Consider the differential equation $\frac{dy}{dx} = \frac{xy}{2}$.

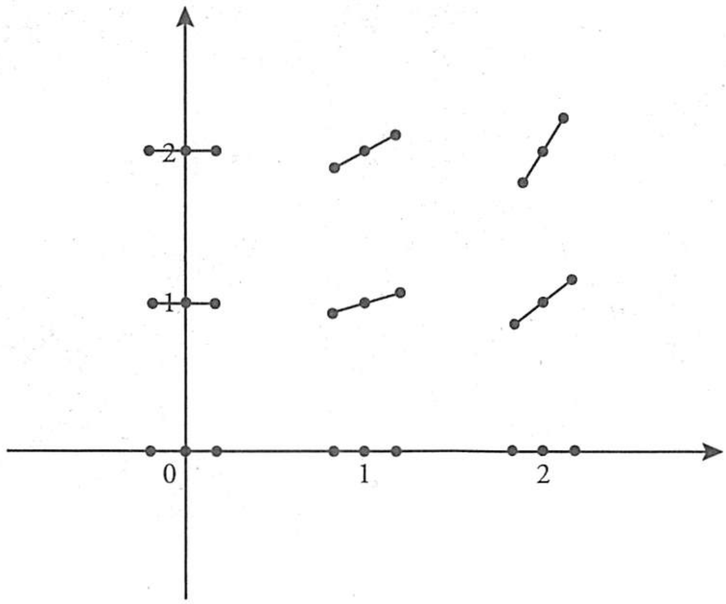
(a) Sketch the slope field at the 9 points indicated below.



Let's make a table of the slopes at each of the points:

x	y	$\frac{dy}{dx} = \frac{xy}{2}$
0	0	0
1	0	0
2	0	0
0	1	0
1	1	$\frac{1}{2}$
2	1	1
0	2	0
1	2	1
2	2	2

Now we will graph a line segment with the slope at each of the points.



(b) Find the particular solution to the differential equation $y = f(x)$ with initial condition $f(0) = 5$.

We can solve this differential equation using Separation of Variables. Divide both sides by y and multiply both sides by dx: $\frac{dy}{y} = \frac{xdx}{2}$. Integrate both sides: $\int \frac{dy}{y} = \int \frac{xdx}{2}$. $\ln|y| = \frac{x^2}{4} + C$. Exponentiate both sides: $y = e^{\frac{x^2}{4}+C} = e^{\frac{x^2}{4}}e^C = Ce^{\frac{x^2}{4}}$. Now solve for C using the initial condition: $5 = Ce^0 = C$. Therefore, the solution is $y = 5e^{\frac{x^2}{4}}$.

(c) Write an equation for the tangent line to the graph at (2, 1). Use the tangent line to estimate y at $x = 2.1$.

The slope of the tangent line is $\frac{dy}{dx} = \frac{(2)(1)}{2} = 1$, so the equation of the tangent line is $y - 1 = 1(x - 2)$, or $y = x - 1$. Therefore, at $x = 2.1$, $y = 2.1 - 1 = 1.1$.