A. Horizontal Asymptotes

What you are finding: Typically, a horizontal asymptote (H.A.) problem is asking you find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

How to find it: Express the function as a fraction. If both numerator and denominator are polynomials,

a) If the higher power of x is in the denominator, the H.A. is $y = 0$.
b) If both the numerator and denominator have the same highest power, the H.A. is the ratio of the coefficients of the highest power term in the numerator and the coefficient of the highest power term in the denominator.
c) If the higher power of x is in the numerator, there is no H.A.

Note that this type of problem hasn't been asked in a free response question since 1995. It is taught in precalculus but teachers incorporate the concepts of limits in calculus to review the concept.

Note: L'Hopital's rule can be used to find these limits if students have learned it. L'Hopital's rule is not in the AB curriculum. I recommend teaching it at the end of the year if there is time.

Example 1: Given $f(x) = \dfrac{6x+1}{\sqrt{4x^2+6x+9}}$, write an equation for any horizontal asymptotes of $f(x)$.

Example 2: Show that $f(x) = \dfrac{\sin x}{e^x}$ has a horizontal asymptote on one side of the y-axis but not on the other side.

B. Approximate Rate of Change

What you are finding: An approximation of the derivative of a function. Typically this type of question occurs when you have a table of values and not a specific function.

How to find it: If you are given a table of n values.

x	x_1	x_2	...	x_i	...	x_{n-1}	x_n
y	y_1	y_2	...	y_i	...	y_{n-1}	y_n

To find the approximate rate of change at $x = i$, you can either use:

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} \text{ or } \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \text{ or } \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}}$$

Graphically, you are finding the slope of the secant line between two points.

Example 3: a) While cruising Glacier Bay in Alaska, Lisa took a lot of pictures. The total number of pictures she took is given by a differentiable function L of time t. A table of selected values of L for the time interval $0 \le t \le 2.5$ is below. Use data from the table to approximate $L'(1.7)$. Show the computation that leads to your answer, explain the meaning of your answer and specify units of measure.

t (hours)	0.0	0.5	1.0	1.5	2.0	2.5
$L(t)$ (pictures)	0	20	45	75	90	100

b) Ted also took a lot of pictures. His rate of picture taking in pictures per hour during the cruise is given by a differentiable function T of time t. A table of selected values of T for the time interval $0 \le t \le 2.5$ is below. Use data from the table to approximate $T'(1.7)$. Show the computation that leads to your answer, explain the meaning of your answer and specify units of measure.

t (hours)	0.0	0.5	1.0	1.5	2.0	2.5
$T(t)$ (pictures per hour)	40	28	53	55	45	24

Note the subtle differences between a) and b). In part a), you are given the function that represents the number of pictures that have been taken. In part b) you are given the rate of pictures taken. In part b), the function was described as $T(t)$ but since it is a rate, it could just as easily be described as $T'(t)$. That doesn't change the units in your answer. In part b), we can also use the 2nd FTC/ accumulation function to estimate the number of pictures Ted took (Section O).

Example 4: The velocity of a particle moving along the y-axis is modeled by a differentiable function v, where the position y is measured in feet and time t is measured in seconds. Selected values of $v(t)$ and $a(t)$ - the acceleration of the particle are given in the table below.

t	0	10	18	30	35	50
$v(t)$	4	12	9	-20	-6	2
$a(t)$	2	3	-1	Unknown	-3.5	3

a) Approximate the acceleration at $t = 15$ sec. Show your computation and indicate units.

b) Approximate the acceleration at $t = 30$ sec. Show your computation and indicate units.

Students can get confused with the words "average rate of change." This means the average value of the rate of change. Typically, this involves finding the average value of a function, which involves the 2nd Fundamental Theorem of Calculus. Section R in this manual stresses the difference between average value as opposed to average rate of change, but here is a problem that illustrates this concept.

Example 5 (Calc): Let $F(x) = \int_0^x \cos(t^2 + t + 1)\, dt$.

a) Find the average rate of change of F on [3,7].

b) Find the average rate of change of $F'(x)$ on [3,7].

C. Intermediate Value Theorem (IVT)

What it says: If you have a continuous function on $[a,b]$ and $f(b) \neq f(a)$, the function must take on every value between $f(a)$ and $f(b)$ at some point between $x = a$ and $x = b$. For instance, if you are on a road traveling at 40 mph and a minute later you are traveling at 50 mph, at some time within that minute, you must have been traveling at 41 mph, 42 mph, and every possible value between 40 mph and 50 mph.

Example 6: A truck travels along a straight track. During the time interval $0 \leq t \leq 30$ seconds, the truck's velocity in meters per second is a differentiable function. The table below shows selected values of this function.

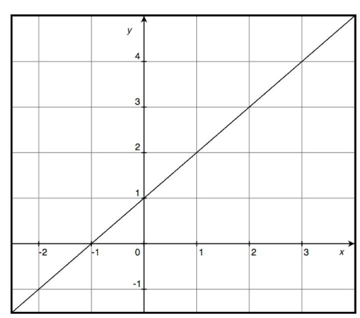
t (sec)	0	5	8	12	20	24	30
$v(t)$ (m/sec)	-10	-25	-5	2	8	-1	3

For $0 \leq t \leq 30$, what is the minimum number of times that the truck must have been stopped? Justify your answer.

Example 7: The functions f and g are continuous. The function h is given by $h(x) = f(g(x)) - x$. The table below gives values of the functions. Explain why there must be a value t for $1 < t < 4$ such that $h(t) = -1$.

x	1	2	3	4
$f(x)$	0	8	-3	6
$g(x)$	3	7	-1	2

Example 8: The straight-line function f is shown by the graph to the right. Define $F(x) = \int_{-2}^{x} f(t)\, dt$. Explain why there must be a value x between 0 and 2 such that $F(x) = \pi$.

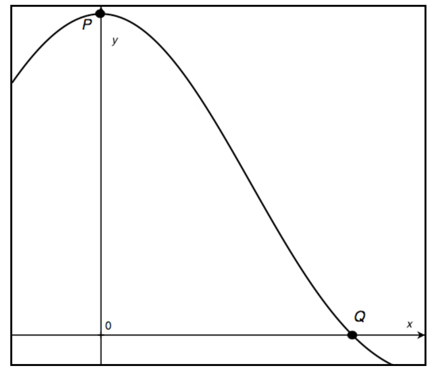


D. Equation of a Tangent Line

What you are finding: You typically have a function f and you are given a point on the function. You want to find the equation of the tangent line to the curve at that point.

How to find it: You use your point-slope equation: $y - y_1 = m(x - x_1)$ where m is the slope and (x_1, y_1) is the point. Typically, to find the slope, you take the derivative of the function at the specified point: $f'(x_1)$.

Example 9: Let f be the function given by $f(x) = \cos^2 x + \cos x$ as shown in the graph to the right. The curve crosses the y-axis at point P and the x-axis at point Q. Find the equation of the tangent line to f at Q.

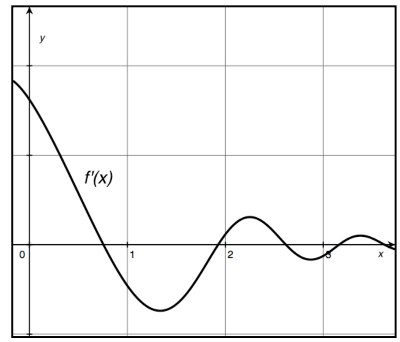


Example 10: Suppose that the function f has a continuous first derivative for all x and that $f(0) = -3$ and $f'(x) = -4$. Let g be a function whose derivative is given by $g'(x) = e^{x-1}(2f(x) - 3f'(x))$ for all x. Given that $g(0) = 2$, write an equation of the tangent line to the graph of g at $x = 0$.

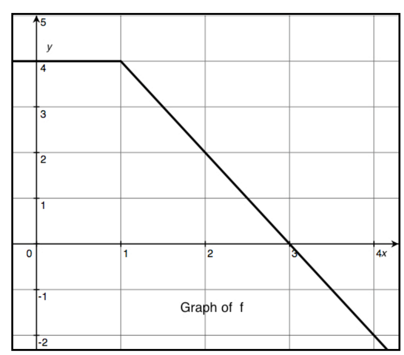
Example 11: Consider the differential equation $\frac{dy}{dx} = \sin^{-1}(x^2 - 2y)$. Let $f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 1$. Write an equation for the tangent line to the graph of f at $x = -1$.

Example 12: Consider the curve in the *xy*-plane $x^3 + 4x^2 + y^2 - 2y = 11$. Write an equation for any lines tangent to the curve when $x = -2$.

Example 13: **(Calc)** Let f be a function defined for $x \ge 0$ with $f(0) = 8$ and f', the derivative of f, is given by $f'(x) = 3e^{-x}\cos(x^2+1)$. The graph of f' is shown to the right. Write an equation of the line tangent to the graph of f at $x = 2$.



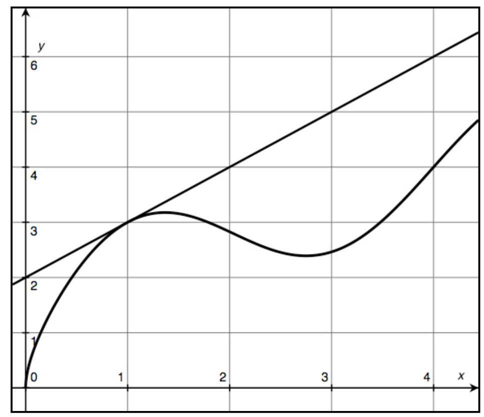
Example 14: The graph of the function f to the right consists of two line segments. Let g be the function $g(x) = \int_0^x f(t)\,dt$. Find the equation of the tangent line to g at $x = 4$.



E. Local Linear Approximations

What you are finding: You typically have a function *f* given as a set of points as well as the derivative of the function at those *x*-values. You want to find the equation of the tangent line to the curve at a value *c* close to one of the given *x*-values. You will use that equation to approximate the *y*-value at *c*. This uses the concept of local linearity – the closer you get to a point on a curve, the more the curve looks like a line.

How to find it: You use your point-slope equation: $y - y_1 = m(x - x_1)$ where *m* is the slope and (x_1, y_1) is the point closest to *c*. Typically, to find the slope, you take the derivative $f'(x_1)$ of the function at the closest *x*-value given. You then plug *c* into the equation of the line. Realize that it is an approximation of the corresponding *y*-value. If 2nd derivative values are given as well, it is possible to determine whether the approximated *y*-value is above or below the actual *y*-value by looking at concavity. For instance, if we wanted to approximate $f(1.1)$ for the curve in the graph to the right, we could find $f'(x)$, use it to determine the equation of the tangent line to the curve at $x = 1$ and then plug 1.1 into that linear equation. If information were given that the curve was concave down, we would know that the estimation over-approximated the actual *y*-value.



Example 15: Let *f* be a function that is differentiable for all real numbers. The table below gives the values of *f* and its derivative f' for selected values in the interval $-0.9 \le x \le 0.9$. The second derivative f'' is always positive in the same closed interval. Write an equation of the line tangent to the graph of *f* where $x = -0.6$. Use this line to approximate the value of $f(-0.5)$. Is this approximation greater or less than the actual value of $f(-0.5)$ and give a reason.

x	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
$f(x)$	-34	-87	-99	-100	-84	-51	21
$f'(x)$	-69	-30	-9	0	1	9	90

Example 16: Consider the curve $x^2 + \ln\left(\frac{x}{y}\right) = 1$. There is a number *k* such that the point (1.25, *k*) is on the curve. Using the tangent line to the curve at $x = 1$, approximate the value of *k*.

F. Continuity and Differentiability

What you are finding: Typical problems ask students to determine whether a function is continuous and/or differentiable at a point. Most functions that are given are continuous in their domain, and functions that are not continuous are not differentiable. So functions given usually tend to be piecewise and the question is whether the function is continuous and also differentiable at the x-value where the function changes from one piece to the other.

How to find it: Continuity: I like to think of continuity as being able to draw the function without picking your pencil up from the paper. But to prove continuity at $x = c$, you have to show that $\lim_{x \to c} f(x) = f(c)$. Usually you will have to show that $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$.

Differentiability: I like to think of differentiability as "smooth." At the value c, where the piecewise function changes, the transition from one curve to another must be a smooth one. Sharp corners (like an absolute value curve) or cusp points mean the function is not differentiable there. The test for differentiability at $x = c$ is to show that $\lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x)$. So if you are given a piecewise function, check first for continuity at $x = c$, and if it is continuous, take the derivative of each piece, and check that the derivative is continuous at $x = c$. Lines, polynomials, exponentials and sine and cosines curves are differentiable everywhere.

Example 17: Let $f(x) = \begin{cases} -6x^2 + 14x - 8, x \le 1 \\ \sin(2x - 2), x > 1 \end{cases}$. Show that $f(x)$ is differentiable.

Example 18: Find the values of a and b that make the following function differentiable:

$$y = \begin{cases} ax^3 - 2, x \le 3 \\ b(x-2)^2 + 10, x > 3 \end{cases}$$

G. Mean Value Theorem (MVT)

What it says: If $f(x)$ is continuous on $[a,b]$ and differentiable on (*a, b*), there must be some value of *c* between *a* and *b* such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. In words, this says that there must be some value between *a* and *b* such that that the tangent line to the function at that value is parallel to the secant line between *a* and *b*.

Example 19: The functions *f* and *g* are continuous and differentiable for all numbers. The function *h* is given by $h(x) = f(g(x)) - x$. The table below gives values of the functions. Explain why there must be a value *t* for $1 < t < 4$ such that $h'(t) = 3$.

x	1	2	3	4
$f(x)$	0	8	-4	6
$g(x)$	3	7	-1	2

Example 20: An amusement park ride travels on a straight track. During the time interval $0 \le t \le 45$ seconds, the vehicle's velocity *v*, measured in meters/sec, and acceleration *a*, measured in meters per second per second shows selected values of these functions.

t	0	10	20	25	30	40	45
$v(t)$ (m/sec)	-4.5	-6.2	-7.3	-6.2	-3.0	0	3.0
$a(t)$ (m/sec^2)	2.2	2.5	1.8	0.7	2.2	2.5	1.5

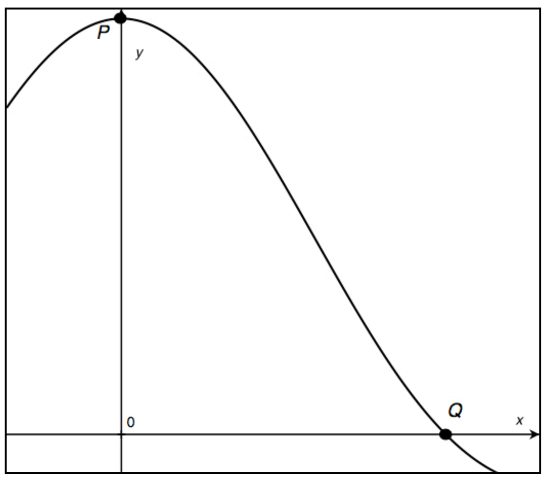
a) For $0 \le t \le 45$, must there be a time *t* when the vehicle has no acceleration? Justify answer.

b) For $0 \le t \le 45$, what is the minimum number of times the vehicle is stopped? Explain.

c) What is the minimum number of times the speed of the vehicle is 1 m/sec? Explain.

Example 21: Let g be a twice-differentiable function such that $g(-1) = 9$ and $g(9) = -1$. Let h be the function such that $h(x) = g(g(x))$. Show that $h'(-1) = h'(9)$. Use this result to show why there must be a value c for $-1 < c < 9$ such that $h''(c) = 0$.

Example 22: (Calc) Let f be the function given by $f(x) = \cos^2 x + \cos x$ as shown in the graph below. The curve crosses the y-axis at point P and the x-axis at point Q. Find the value of c guaranteed by the Mean Value Theorem on [0, Q].



H. Horizontal and Vertical Tangent Lines

How to find them: You need to work with $f'(x)$, the derivative of function *f*. Express $f'(x)$ as a fraction.

Horizontal tangent lines: set $f'(x)=0$ and solve for values of *x* in the domain of *f*.

Vertical tangent lines: find values of *x* where $f'(x)$ is undefined (the denominator of $f'(x)=0$).

In both cases, to find the point of tangency, plug in the *x* values you found back into the function *f*. However, if both the numerator and denominator of $f'(x)$ are simultaneously zero, no conclusion can be made about tangent lines. These types of problems go well with implicit differentiation.

Example 23: (Calc) Let $f''(x)=\dfrac{x}{x^2+1}$ with $f'(0)=-1$. Find the number of horizontal tangent lines the graph of $f(x)$ has on $-5 \le x \le 5$.

Example 24: Let $x^2-2x+y^4+4y=23$. Find the equation of the line(s) that are vertically tangent to $f(x)$.

Example 25: Consider the curve $y^3+2x^2y-2x^2+2y+12=0$. Find the equation of the line(s) that are horizontally tangent to $f(x)$.

I. Related Rates

This is a difficult topic to represent. There are many types of questions that could be interpreted as related rates problems. It can be argued that straight-line motion problems fall under the category of related rates. So do the types of questions that interpret a derivative as a rate of change. In this section, we will only focus on questions that use geometric shapes with different quantities changing values.

What you are finding: In related rates problems, any quantities, given or asked to calculate, that are changing are derivatives with respect to time. If you are asked to find how a volume is changing, you are being asked for $\frac{dV}{dt}$. Typically in related rates problems, you need to write the formula for a quantity in terms of a single variable and differentiate that equation with respect to time, remembering that you are actually performing implicit differentiation. If the right side of the formula contains several variables, typically you need to link them somehow. Constants may be plugged in before the differentiation process but variables that are changing may only be plugged in after the differentiation process.

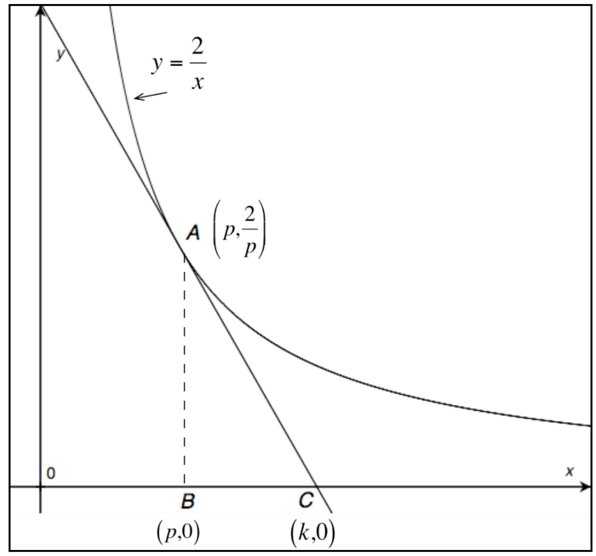
Example 26: Oil is leaking from an oil well at the bottom of a 300-foot deep lake and forms an oil slick that takes the form of a right circular cone. The radius of the slick on the lake's surface is increasing at 10 ft/min at the moment the oil slick is 100 feet wide.

a) Find the rate at which the area of slick is increasing at that time. Indicate units.

b) Find the rate at which the volume of the slick is changing and express your answer in the proper units. (The volume V of a cone with radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$)

c) A special ship that skims the oil and removes it is dispatched to the scene and starts to remove oil at the moment that the radius of the slick was 100 feet wide. The rate at which oil is removed is $S(t) = 10000\pi\sqrt[3]{t}\ \frac{\text{ft}^3}{\text{min}}$ where t is the time in minutes since the ship starts to remove oil. Using the value you found in part b) as the rate that oil is escaping the well, find the time t when the oil slick reaches its maximum volume.

Example 27: In the figure to the right, the line is tangent to the graph of $y = \frac{2}{x}$ at point A with coordinates $\left(p, \frac{2}{p}\right)$. Point B has coordinates $(p, 0)$. The line crosses the x-axis at the point C with coordinates $(k, 0)$.



a) Find k in terms of p.

b) Suppose p is increasing at the constant rate of 8 units per second. What is the rate of change of k with respect to time?

c) Suppose p is increasing at a constant rate. Show that there is no change to the area of triangle ABC with respect to time.

d) Suppose p is increasing at a constant rate of 4 units/sec. Find the rate of change of hypotenuse AC of the triangle ABC when $p = 2$.

Sometimes, a problem that appears to be a related rates question could be a differential equation in disguise.

Example 28: At time t measured in minutes, the volume of a cube is increasing at a rate proportional to the reciprocal of the length of its side x, measured in inches. At $t = 0$, the side of the cube is 2 inches and at $t = 2$, the side of the cube is 4 inches.

a. Find the side of the cube in terms of t.

b. At what time t will the volume of the cube be 125?

c. When the volume of the cube is 125 and growing at $150\dfrac{\text{in}^3}{\text{min}}$, how fast are the sides growing at that time?

J. Straight-Line Motion – Derivatives

What you are finding: Typically in these problems, you work your way from the position function $x(t)$ to the velocity function $v(t)$ to the acceleration function $a(t)$ by the derivative process. Finding when a particle is stopped involves setting the velocity function $v(t)=0$. Speed is the absolute value of velocity. Determining whether a particle is speeding up or slowing down involves finding both the velocity and acceleration at a given time. If $v(t)$ and $a(t)$ have the same sign, the particle is speeding up at a specific value of *t*. If their signs are different, then the particle is slowing down.

Example 29: (Calc) A particle's movement along the *x*-axis at time $t \geq 0$ is given by $v(t) = 3 - 4\tan^{-1}\left(e^{t^2-1}\right)$.

a) Find the acceleration of the particle at time $t = 1$.

b) Is the speed of the particle increasing or decreasing at time $t = 1$. Explain your answer.

c) Find the time $t \geq 0$ at which the particle is farthest to the right. Justify your answer.

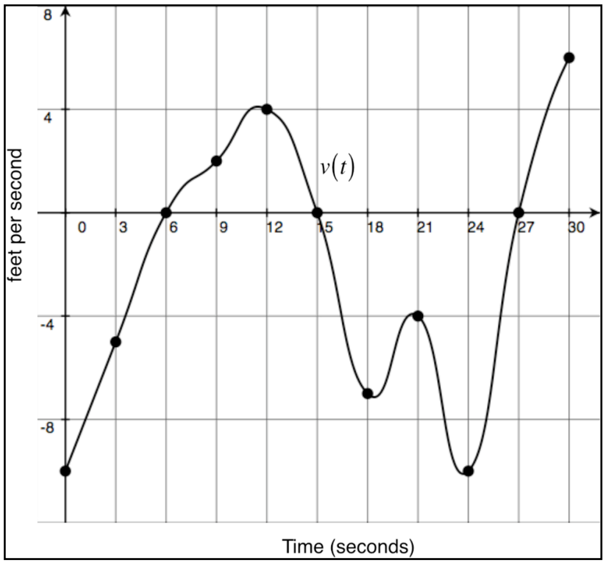
Example 30: A particle moves along the *y*-axis with position at time $t \geq 0$ given by $y(t) = \cos\left(\frac{4\pi}{3}t\right)$.

a) Find the acceleration of the particle at time $t = 3$.

b) Determine if the velocity of the object is decreasing at $t = 3$. Explain your answer.

c) Determine if the speed of the object is decreasing at $t = 0.5$. Explain your answer.

Example 31: A searchlight is shining along the straight wall of a prison. A graph of the velocity of the light, $v(t)$ at 3-second intervals of time t is shown in the table as well as a table of values.



t (seconds)	$v(t)$ ft per second
0	-10
3	-5
6	0
9	2
12	4
15	0
18	-7
21	-4
24	-10
27	0
30	6

a) At what interval of times is the acceleration of the searchlight positive? Why?

b) Find the average acceleration of the searchlight over the interval $9 \le t \le 27$. Express units.

c) Give an approximation for the acceleration of the searchlight at $t = 15$. Show the computation you used to arrive at your answer.

d) For what values of t is the searchlight speeding up on the wall? Justify your answers.

K. Function Analysis

What you are finding: You have a function $f(x)$. You want to find intervals where $f(x)$ is increasing and decreasing, concave up and concave down. You also want to find values of *x* where there is a relative minimum, a relative maximum, and points of inflection.
How to find them: Find critical values – values $x = c$ where $f'(c) = 0$ or $f'(c)$ is undefined.

$f(x)$ is increasing for values of *k* such that $f'(k) > 0$

$f(x)$ is decreasing for values of *k* such that $f'(k) < 0$.

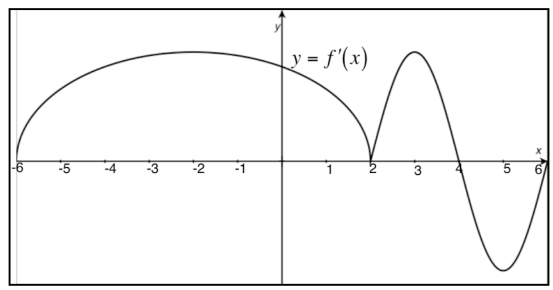
If $f(x)$ switches from increasing to decreasing at *c*, there is a relative maximum at $(c, f(c))$. If $f(x)$ switches from decreasing to increasing at *c*, there is a relative minimum at $(c, f(c))$. This is commonly called the first derivative test.

$f(x)$ is concave up for values of *k* such that $f''(k) > 0$.

$f(x)$ is concave down for values of *k* such that $f''(k) < 0$.

If $f(x)$ switches concavity at *c*, there is a point of inflection at $(c, f(c))$. If you want to find the actual maximum or minimum value or find the value of the function at specific points, you need to use accumulated area and the FTC.

Example 32: The figure to the right shows the graph of f', the derivative of the function *f*, for $-6 \le x \le 6$. The graph of f' has horizontal tangents at $x = -2$, $x = 3$, and $x = 5$ and is symmetric with respect to the *x*-axis from $x = 2$ to $x = 6$.



a) Find all values of *x*, for $-6 < x < 6$, at which *f* attains a relative maximum. Justify answer.

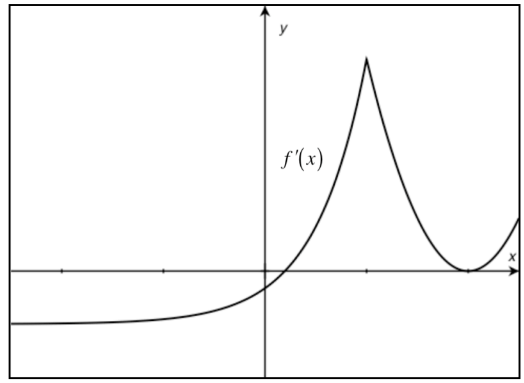
b) Find all values of *x*, for $-6 < x < 6$, at which *f* attains a relative minimum. Justify answer.

c) Find all values of *x*, for $-6 < x < 6$, at which *f* has an inflection point. Justify answer.

d) Find the value of *x* for which *f* has an absolute minimum. Justify answer.

Example 33: (Calc) The derivative of a function *f* is defined by

$$f'(x)=\begin{cases}5e^{2x-2}-1 & \text{for } -2.5\le x<1\\ 4(x-2)^2 & \text{for } 1\le x\le 2.5\end{cases}$$



The graph of the continuous function f' is shown in the figure on the right. $f(0)=-2$.

a) For what value of x does the graph of f have a point of inflection. Justify your answer.

b) For what values of x is f decreasing on [-2.5,2.5] ? Justify your answer.

c) For what value of x does f attain an absolute minimum on [-2.5,2.5].

d) What is the absolute minimum of f on [-2.5,2.5]?

e) Find $f(-2.5)$ and $f(2.5)$

L. Second Derivative Test

What you are finding: Students normally find points of relative maximum and relative minimum of $f(x)$ by using the first derivative test, finding points where a function switches from increasing to decreasing or vice versa. But students may be forced to use the 2nd derivative test, as seen below.

How to use it: Find critical values c where $f'(c)=0$.

a) If $f''(c)>0$, f is concave up at c and there is a relative minimum at $x=c$.

b) If $f''(c)<0$, f is concave down at c and there is a relative maximum at $x=c$.

c) If $f''(c)=0$, the 2nd derivative test is inconclusive. $\left(\text{ex. } f(x)=x^4 \text{ at } x=0\right)$

Example 34: Let f be the function defined by $f(x)=k\sqrt[3]{x}-e^x$ for $x \geq 0$, where k is a constant.

a) For what value of k does f have a critical point at $x = 1$?

b) For the value of k in part *a*), determine whether f has a relative maximum, relative minimum, or neither at $x = 1$. Justify your answer.

Example 35: Consider the differential equation $\dfrac{dy}{dx}=2x+\dfrac{y}{2}-4$.

a) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave down.

b) Let $y=f(x)$ be a particular solution to the differential equation with the initial condition $f(0)=8$. Does f have a relative minimum, relative maximum or neither at $x = 0$? Justify.

M. Absolute Extrema

What you are finding: the highest value or lowest value of a function, typically in an interval. Another way of asking the question is to ask for the range of the function.

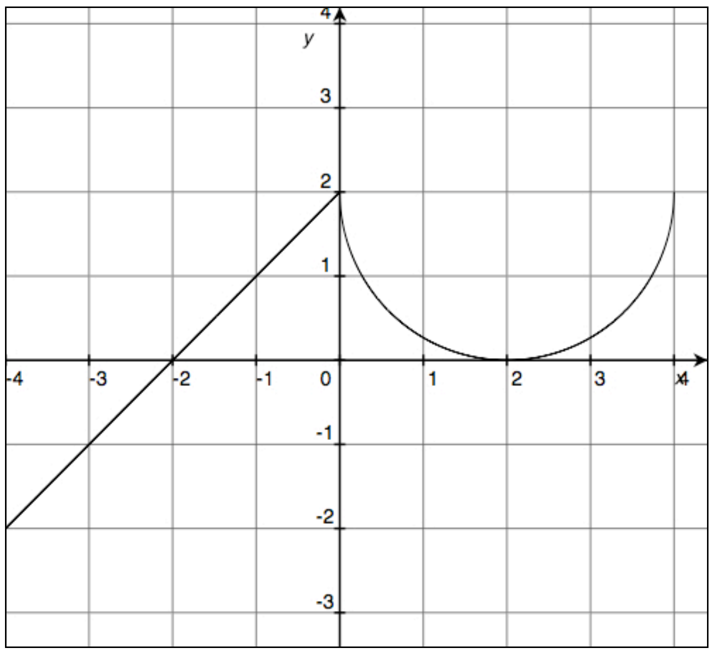
How to find it: a) find critical points (x-values where the derivative = 0 or is not defined).
b) use 1st derivative test to determine x-values when the function is increasing/decreasing.
c) evaluate function at critical points and at the endpoints. This may involve using the FTC to find accumulated change.

Example 36: (Calc) A particle moves along the y-axis such that its velocity v is given by $v(t) = 1 - 2\sin^{-1}\left(e^{-t}\right)$ for $0 \le t \le 2$. Find the time when it reaches its lowest point.

Example 37: Find the range of $f(x) = \dfrac{4x - 4}{\sqrt{x^2 - x + 2}}$.

Example 38: The derivative of a function f is defined by $f'(x)=\begin{cases} x+2 & -4 \le x < 0 \\ g(x) & 0 \le x \le 4 \end{cases}$ where $g(x)$ is a semicircle of radius 2 and $f(0)=8$. The graph of f' is below.

For $-4 \le x \le 4$, find the minimum and maximum value of f.



Note that this problem incorporates both the concept of accumulation as well as integration of the rate of change to give accumulated change.

Example 39: A 3-hour concert serves hot dogs as refreshments. The number of hot dogs that are cooked is modeled by a continuous function on the time interval $0 \le t \le 3$. In this model, rates are given as follows:

(i) the rate at which hot dogs are cooked is $f(t) = 10 - t^2$, measured in hundreds.
(ii) the rate at which hot dogs are sold is $g(t) = t^2 + t$, measured in hundreds.

When hot dogs are cooked, they are placed in warmers.
There are 500 hot dogs already in the warmers at the time the doors open ($t = 0$).

a) For $0 \le t \le 3$, at what time t is the number of hot dogs that are either on the grill or are in the warmers at a maximum? (hot dogs that have cooked or are cooking but not sold)

b) The concert goes 45 minutes longer than originally planned. No hot dogs are cooked during that time but the rate of hot dogs sold continues to be $g(t)$. Determine how many hot dogs will be left unsold.

N. Computation of Riemann Sums

What you are finding: Riemann sums are approximations for definite integrals, which we know represent areas under curves. There are numerous real-life models for areas under curves so this is an important concept. Typically these types of problems show up when we are given data points as opposed to algebraic functions.

How to find them: Given data points:

x	x_0	x_1	x_2	...	x_{n-2}	x_{n-1}	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	...	$f(x_{n-2})$	$f(x_{n-1})$	$f(x_n)$

Assuming equally spaced x-values: $x_{i+1} - x_i = b$

Left Riemann Sums: $S = b(x_0 + x_1 + x_2 + ... + x_{n-1})$

Right Riemann Sums: $S = b(x_1 + x_2 + ... + x_{n-1} + x_n)$

Trapezoids: $S = \frac{b}{2}(x_0 + 2x_1 + 2x_2 + ... + 2x_{n-2} + 2x_{n-1} + x_n)$

If bases are not the same (typical in AP questions), you have to compute the area of each trapezoid: $\frac{1}{2}(x_{i+1} - x_i)\left[f(x_i) + f(x_{i+1})\right]$

Midpoints: This is commonly misunderstood. For example, you cannot draw a rectangle halfway between $x = 2$ and $x = 3$ because you may not know $f(2.5)$. You can't make up data. So in the table above the first midpoint rectangle would be drawn halfway between x_0 and x_2 which is x_1. So assuming that the x-values are equally spaced, the midpoint sum would be $S = 2b(x_1 + x_3 + x_5 + ...)$.

Example 40: (Calc) James is doing some biking. His velocity in feet/minute is given by a differentiable function V of time t. The table to the right shows the velocity measured every 3 minutes for a 30-minute period.

t (minutes)	$v(t)$ ft per minute
30	880
33	1,056
36	1,408
39	440
42	616
45	836
48	264
51	550
54	610
57	1,200
60	620

a) Find the difference in the estimation of $\int_{30}^{60} V(t)\,dt$ using left Riemann sums and right Riemann sums.

b) Use a midpoint Riemann sum with 5 subdivisions to approximate $\frac{1}{5280}\int_{30}^{60} V(t)\,dt$. Using correct units, explain the meaning of your answer.

Example 41: A weather balloon travels along a straight line vertically. During the time interval $0 \le t \le 60$ seconds, the balloon's velocity v, measured in meters/sec, and acceleration a, measured in meters per second per second, are continuous functions. The table below shows selected values of these functions.

t (sec)	0	5	15	30	40	55	60
$v(t)$ m/sec	-30	-40	-20	-5	0	20	10
$a(t)$ m/sec^2	2	4	1	1	2	5	3

a) Approximate $\int_0^{60} |v(t)|\, dt$ using a right Riemann sum and explain its meaning in terms of the balloon's motion using appropriate units.

b) Approximate $\int_0^{60} a(t)\, dt$ using a trapezoidal approximation with three subintervals and explain its meaning in terms of the balloon's motion using appropriate units.

c) Find the exact value of $\int_0^{60} a(t)\, dt$.

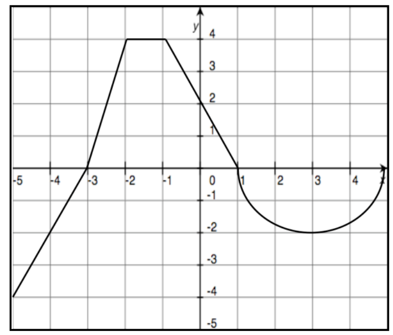
Example 42: (Calc) Let $F(x) = \int_1^x \sin(\ln x)\, dx$. Use the trapezoidal rule with four equal subdivisions to approximate $F(2)$.

O. Accumulation Function / Fundamental Theorem of Calculus (FTC)

What you are finding: the accumulation function looks like this: $F(x) = \int_0^x f(t)\,dt$. It represents the accumulated area under the curve f starting at zero (or some value) and going out to the value of x. The variable t is a dummy variable. It is important to believe that this is a function of x.

How to find it: Typically this is used with particle motion. When f represents a velocity, the accumulation function will represent the displacement of the particle from time = 0 to time = x (section S in this manual). This type of function goes hand-in-hand with absolute max/min problems.

Example 43: A graph of f is made up of lines and a semi-circle as shown to the right. Let $F(x) = \int_{-1}^{x} f(t)\,dt$. Find $F(5)$ and $F(-5)$.



Example 44: The table below gives values of the continuous velocity function of an elevator in a building at selected times. The elevator is 40 feet high at time $t = 0$.

t (seconds)	0	20	30	50	70	100	120	130	160
$v(t)$ (ft/sec)	6	8	-2	4	0	-5	-1	2	3

a. Use a right Riemann sum with 4 intervals to approximate $40 + \int_0^{120} v(t)\,dt$ and explain its meaning. Show your calculations.

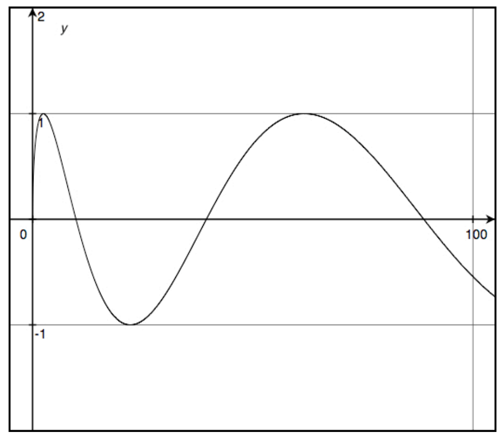
b. Use a trapezoidal approximation with 5 trapezoids to approximate $\int_{20}^{120} |v(t)|\,dt$ and explain its meaning. Show your calculations.

c. Using the table above, what is the minimum number of values of t such that $\frac{d}{dt}\int_0^t v(k)\,dk = 0$? Justify your answer.

This type of problem is fair game to show up on an exam. It illustrates the meaning of the Fundamental Theorem of Calculus.

Example 45: (Calc) $F(x) = \int 2^{x^2-2x+1}dx$ and $F(1) = 2$, find $F(2)$.

Example 46: A particle moves along the x-axis such that its velocity v at time $t \geq 0$ is given by $v(t) = \sin\sqrt{t}$ for $0 \leq t \leq 100$. The graph of $v(t)$ is shown to the right. The position of the particle at time t is $x(t)$ and its position at time $t = 0$ is $x(0) = 10$. Find the times when the particle is farthest to the right and farthest to the left. Explain your answer.



P. Interpretation of a Derivative as a Rate of Change

What you are finding: As mentioned in section I (Related Rates), a quantity that is given as a rate of change needs to be interpreted as a derivative of some function. Typical problems ask for the value of the function at a given time. These problems can be handled several ways:

a) solving a Differential Equation with initial condition (although DEQ's may not have even been formally mentioned yet)

b) Integral of the rate of change to give accumulated change. This uses the fact that:

$$\int_a^b R'(t)\,dt = R(b) - R(a) \text{ or } R(b) = R(a) + \int_a^b R'(t)\,dt$$

Example 47: A car's gas tank contains 4 gallons. A gas pump can fill the tank at the rate of $\sqrt{9-t}$ gallons per minute for $0 \le t \le 10$ minutes. How many gallons of gas are in the tank at $t = 5$ minutes?

Example 48: (Calc) Frankenstein Electronics makes gold-plated HDMI cables for HD TV's that sell for $9 a foot. Frankenstein says that the cost of creating an x-foot cable is $\dfrac{\sqrt{x^3}}{3}$ dollars.

a) Write an expression involving an integral that represents Frankenstein's profit on a cable of length k.

b) Find the maximum profit that Frankenstein could earn on a cable. Justify your answer.

Example 49: (Calc) The Washington DC subway system uses cards for people to enter the system and for people to leave. So it is known how many people enter and leave the system at any one time. In a certain 12-hour period, people enter the system at a rate modeled by the function *E*.

$$E(t) = 2 + 3\sin\left(\frac{2\pi t}{25}\right)$$

In the same 12-hour period, people leave the system at a rate modeled by the function *L.s*

$$L(t) = -0.11t^2 + 1.41t + 0.57$$

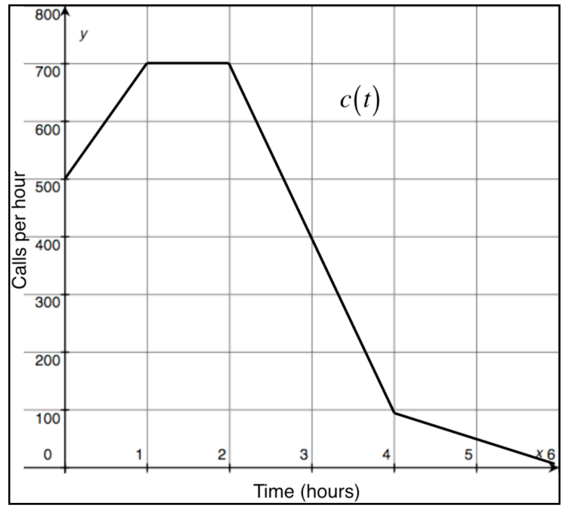
Both $E(t)$ and $L(t)$ are measured in thousands of people per hour and t is measured in hours for $0 \le t \le 12$. At $t = 0$, there are 5,000 people on the subway system.

a) How many more people enter the subway during the second 6 hours than the first 6 hours?

b) Write an expression for $P(t)$, the total number of people on the subway at time t.

c) For what time is the number of people on the subway system a maximum? What is the maximum value? Justify your answers.

Example 50: (Calc) A new Broadway show is opening and people call in for tickets. There are 300 people waiting on hold on the phone when the operators start processing calls. They can handle 400 calls per hour. The graph to the right consisting of four straight lines shows the rate $c(t)$ at which new calls are coming in. Time t is measured in hours from when the processing starts. Assume that no one hangs up when on hold.



a) How many people have called in by the end of the 6-hour period?

b) Is the number of people who are waiting on hold increasing, decreasing, or staying the same between hour 1 and hour 2? Give a reason for your answer.

c) At what time t is the number of people on hold the longest? How many people are on hold at that time?

d) Write but do not solve an equation involving an integral expression of c whose solution gives the earliest time k when there is no one waiting on hold.

Q. Derivative of Accumulation Function (2nd FTC)

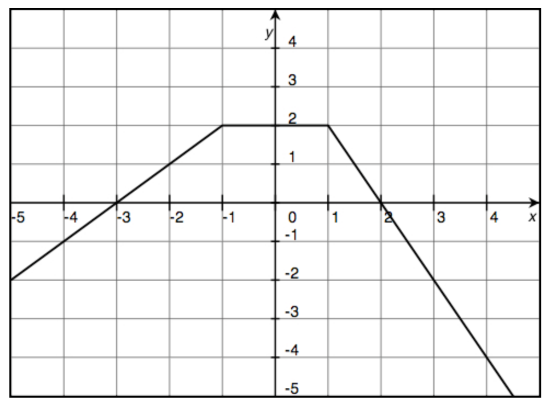
What you are finding: You are looking at problems in the form of $\frac{d}{dx}\int_a^x f(t)\,dt$. This is asking for the rate of change with respect to *x* of the accumulation function starting at some constant (which is irrelevant) and ending at that variable *x*. It is important to understand that this expression is a function of *x*, not the variable *t*. In fact, the variable *t* in this expression could be any variable (except *x*).

How to find it: You are using the 2nd Fundamental Theorem of Calculus that says: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Occasionally you may have to use the chain rule that says $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$.

Example 51: Let $f(x)$ be defined by the graph to the right whose domain is [-5,5].

Let $F(x) = \int_{-3}^{x} f(t)\,dt$ and $F(-3) = 0$



a) Put $F(4), F'(4)$ and $F''(4)$ in order from largest to smallest.

b) Find the equation of the tangent line to *F* at *x* = 4.

c) Use the results of b) to approximate the value of *F* at *x* = 4.1. Does this value over-approximate or under-approximate $F(4.1)$? Justify your answer.

d) Find the value of *x* where *F* has a maximum. Justify your answer.

Example 52: Let $f(x)$ be a function that is continuous on the interval $[-1,3)$. The function is twice-differentiable except at $x = 1$. The function f and its 1st and 2nd derivatives have the properties in the table below, where DNE means that the derivative doesn't exist.

x	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	3	Positive	0	Negative	-5	Negative	0	Negative
$f'(x)$	-10	Negative	0	Negative	DNE	Positive	0	Negative
$f''(x)$	6	Positive	0	Negative	DNE	Negative	0	Negative

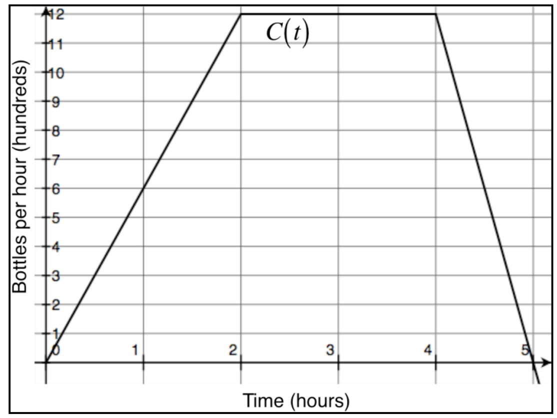
a) Let g be the function $g(x) = \int_2^x f(t)\,dt$ on the interval $(-1,3)$. For $-1 < x < 3$, find all values of x at which g has a potential relative extremum. Classify each as a relative maximum, relative minimum, or neither. Justify your answer.

b) For the function g defined in part a), find all values of x, for $-1 < x < 3$, at which the graph of g has a point of inflection. Justify your answer.

Example 53: The functions f and g are differentiable for all real numbers and g is strictly decreasing. The table below gives values of the functions and their first derivatives at selected values of x. Let h be the function given by $h(x) = \int_{-1}^{g(x)} f(t)\,dt$. Find the value of $h'(-1)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	3	5	4	-3
-1	5	1	0	-2
0	8	-3	-2	-1
1	-2	-4	-3	-2

Example 54: A juice bottling company has new bottles of juice emerge from a sanitizer and placed on a conveyor belt. From the conveyor belt, they are placed into boxes for shipping.



Let $C(t)$, made up of three straight lines shown by the graph on the right, represent the rate, measured in hundreds of bottles per hour that bottles of juice go onto the conveyor belt.

The rate that bottles are boxed is $B(t) = 900$, measured in bottles/hr. At the start of the 5-hour shift, there are 800 bottles on the belt. Suppose $F(t) = \int_0^t \left[C(x) - B(x)\right] dt$.

a) Write and calculate an expression using F that represents the number of bottles on the belt at 2 hours.

b) Find $F'(1)$ and express its meaning.

c) When will the number of bottles on the conveyor belt be at a minimum and how many bottles will be on the belt at that time? Justify your answer.

d) Assume that after the 5-hour shift ends, another 5-hour shift starts up with the same graph of $C(t)$ and the same boxing rate of $B(t)$. Write, but do not solve, an equation in terms of time $t = k$ for when the conveyor belt is empty.

R. Average Value of a Function

What you are finding: You are given a continuous function $f(x)$, either an algebraic formula or a graph, as well as an interval $[a,b]$. You wish to find the average value of the function on that interval.

How to find it: $f_{avg} = \dfrac{\int_a^b f(x)\,dx}{b-a}$. The units will be whatever the function *f* is measured in.

Again, be careful. The average rate of change of a function *F* on $[a,b]$ is not the same as the average value of the function *F* on $[a,b]$.

<u>Average value of the function</u>: $\dfrac{\int_a^b f(x)\,dx}{b-a}$

<u>Average rate of change of the function</u> (average value of the rate of change): $\dfrac{\int_a^b F'(x)\,dx}{b-a} = \dfrac{F(b)-F(a)}{b-a}$

Example 55: If $f(x) = x^2 - 2x + 3$, find

a) the average rate of change of *f* on [-3, 3]:

b) the average value of *f* on [-3, 3]:

Example 56: (Calc) If $f(t) = \ln\left(t^2 + 2t - 2\right)$, find

a) the average rate of change of *f* on [1, *e*]:

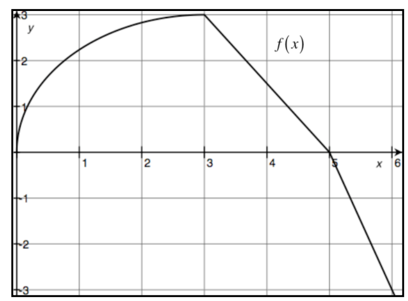
b) the average value of *f* on [1, *e*]:

Example 57: A particle moves along the x-axis such that its velocity is given by $v(t) = t\cos(t^2)$. The particle is at position $x = 3$ at time $t = 0$. Assume v is measured in feet and t is measured in seconds.

a) Find the average speed of the particle over the interval $0 \le t \le \sqrt{\pi/2}$. Specify units.

b) Find the average acceleration of the particle over the interval $0 \le t \le \sqrt{\pi}$.

Example 58: The graph of $f(x)$ is comprised of a quarter circle and 2 lines as shown in the figure on the right.



a) Find the average value of $f(x)$ on [0, 6].

b) Suppose the straight line between $x = 5$ and $x = 6$ in the graph continues. Write, but do not solve, an equation to find the value of k such that the average value of $f(x)$ on [0, k] is zero.

Example 59: The "Laughing Gas" company rents helium tanks in the shape of a cylinder. Their tanks are all 2 feet long but have variable radii, from 2 to 6 inches. The dollar cost of renting a tank is

$$C = 20 + \frac{\text{Volume of cylinder in inches}^3}{100 \text{ inches}^3}$$

a) Find the average cost of all the tanks that the company rents.

b) Find the average rate of change of the cost as one changes from a 2 inch radius tank to a 6 inch radius tank. Use appropriate units.

Average value problems can cause some confusion. Let us look at a slightly different type of problem:

Example 60: Jamie takes on a job to provide four hours of babysitting. She agrees on a price of \$40 for the four hours.

a) What is her average pay per hour?

$$\boxed{\text{Certainly a question that a grade - school student can answer. \$10 an hour.}}$$

Suppose that Jamie is paid at the **end** of the 4 hour session. Let's think of her payment in terms of how much additional money is in her wallet at the end of every hour.

Hour (*t*)	0	1	2	3	4
Money (*M*)	0	0	0	0	40

b) Calculate $\int_0^4 M(t)\,dt$ and $\frac{1}{4}\int_0^4 M(t)\,dt$ using right Riemann sums and interpret their meaning using proper units.

c) Calculate $\int_0^4 M(t)\,dt$ and $\frac{1}{4}\int_0^4 M(t)\,dt$ using left Riemann sums and interpret their meaning using proper units assuming that Jamie gets paid at the **beginning** of the 4-hour session.

Hour (*t*)	0	1	2	3	4
Money (*M*)	40	40	40	40	40

d) Calculate $\int_0^4 M(t)\,dt$ and $\frac{1}{4}\int_0^4 M(t)\,dt$ using right Riemann sums and interpret the meaning using proper units assuming that Jamie gets paid **halfway through** the 4-hour session.

Hour (*t*)	0	1	2	3	4
Money (*M*)	0	0	40	40	40

Example 61: (Calc) A tennis tournament is free for people to watch. People line up at the gate waiting for it to open in order to get the best seats. . The number of people in line is modeled by a differentiable function L for $0 \le t < 60$ where t is measured in minutes. Values of $L(t)$ at various times t are shown in the table below.

t (minutes)	0	10	20	30	45	60
$L(t)$ (people in line)	60	80	120	200	350	600

a) Find the average rate of people lining up from $t = 0$ to $t = 60$. Specify units.

b) Approximate the rate in which people are lining up at $t = 60$ minutes.

c) Using the answer from part b), estimate the number of people in line at $t = 55$ minutes. Show your reasoning.

d) Using a trapezoidal sum with the five subintervals given by the table, approximate the value of $\int_0^{60} L(t)\,dt$. Explain the meaning of $\int_0^{60} L(t)\,dt$ in terms of people in line.

e) Using the answer from part d), approximate the value of $\frac{1}{60}\int_0^{60} L(t)\,dt$. Explain the meaning of $\frac{1}{60}\int_0^{60} L(t)\,dt$ in terms of people in line.

S. Straight-Line Motion - Integrals

What you are finding: In section J, we looked at straight-line motion by the derivative process. You were typically given a position function $x(t)$ and took its derivative to find the velocity function $v(t)$ and took the velocity's derivative to get the acceleration function $a(t)$. Questions like finding maximum velocity or minimum acceleration could then be answered.

Using integrals, we can work backwards. Typical questions involve knowing the velocity function, the position of the particle at the start of the problem, and integrating to find the position function and then finding when the particle is farthest to the right or left.

How to find it: $x(t) = \int v(t)\,dt + C$ and $v(t) = \int a(t)\,dt + C$. Using these equations is essentially solving DEQ's with an initial condition, usually the position at time $t = 0$ or velocity at $t = 0$. Two other concepts that come into play are displacement and distance over some time interval $[t_1, t_2]$.

Displacement: the difference in position over $[t_1, t_2]$: $x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)\,dt$. Displacement can be positive, negative or zero.

Distance: how far the particle traveled over $[t_1, t_2]$: $\int_{t_1}^{t_2} |v(t)|\,dt$. Distance is always positive.

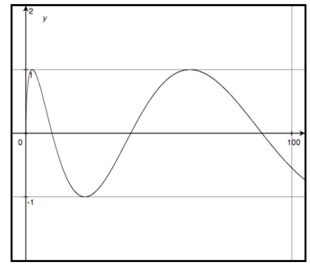
Example 62: A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $3t^2 - 5t - 2$. The position of the particle at time $t = 2$ is $x = 5$.

a) Write a polynomial expression for the position of the particle at any time $t \geq 0$.

b) Find the displacement of the particle from time $t = 0$ until time $t = 4$.

c) Find the distance that the particle moves from time $t = 0$ until time $t = 4$.

Example 63: (Extension of Ex. 46) (Calc) A particle moves along the x-axis such that its velocity v at time $t \geq 0$ is given by $v(t) = \sin\sqrt{t}$ for $0 \leq t \leq 100$. The graph of $v(t)$ is shown to the right. The position of the particle at time t is $x(t)$ and its position at time $t = 0$ is $x(0) = 10$.



a) Find the distance traveled by the particle from $t = 0$ to $t = 100$.

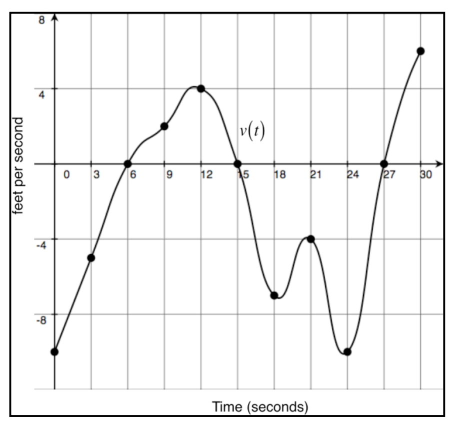
b) Find the position of the particle at time $t = 100$.

Example 64: (Extension of Ex. 31) A searchlight is shining along the straight wall of a prison. A graph of the velocity of the light, $v(t)$ at 3-second intervals of time t is shown in the table as well as a table of values.

t (seconds)	$v(t)$ ft per second
0	-10
3	-5
6	0
9	2
12	4
15	0
18	-7
21	-4
24	-10
27	0
30	6

a) Using a midpoint Riemann sum with 5 rectangles, approximate the position of the searchlight at $t = 30$ seconds from where it was at $t = 0$.

b) Using the trapezoidal method with 10 trapezoids, approximate how far the searchlight traveled in 30 seconds.



T. Area/Volume Problems

What you are finding: Typically, these are problems with which students feel more comfortable because they are told exactly what to do or to find. Area and volume are lumped together because, almost always, they are both tested within the confines of a single A.P. free response question. Usually, but not always, they are on the calculator section of the free-response section.

Area problems usually involve finding the area of a region under a curve or the area between two curves between two values of x. Volume problems usually involve finding the volume of a solid when rotating a curve about a line.

How to find it: Area: Given two curves $f(x)$ and $g(x)$ with $f(x) \geq g(x)$ on an interval $[a, b]$, the area between f and g on $[a, b]$ is given by $A = \int_{x=a}^{x=b} \left[f(x) - g(x)\right] dx$. While integration is usually done with respect to the x-axis, these problems sometimes show up in terms of y; $A = \int_{y=c}^{y=d} \left[m(y) - n(y)\right] dy$.

Volume: Disks and Washers: The method I recommend is to establish the outside Radius R, the distance from the line of rotation to the outside curve, and, if it exists, the inside radius r, the distance from the line of rotation to the inside curve. The formula when rotating these curves about a line on an interval is given by:

$$V = \pi \int_{x=a}^{x=b} \left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right) dx \quad \text{or} \quad V = \pi \int_{y=c}^{y=d} \left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy$$

A favorite type of problem is creating a solid with the region R being the base of the solid. Cross sections perpendicular to an axis are typically squares, equilateral triangles, right triangles, or semi-circles. Rather than give formulas for this, it is suggested that you draw the figure, establish its area in terms of x or y, and integrate that expression on the given interval.

Example 65: (Calc) Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$.

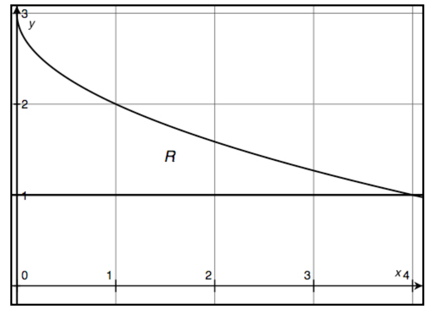
a) Find the area of R.

b) Find the volume of the solid when R is rotated about the x-axis.

c) Find the volume of the solid when R is rotated about the y-axis.

d) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Find the volume of this solid.

Example 66: Let R be the region in the first quadrant bounded by the graphs of $y = 3 - \sqrt{x}$, the horizontal line $y = 1$, and the y-axis as shown in the figure to the right.

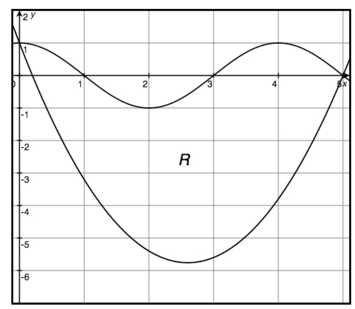


a) Find the area of R.

b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -1$.

c) Region R is the base of a solid. For each y, where $1 \le y \le 3$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is half the length of its base. Write, but do not evaluate, an integral expression that gives the volume of the solid.

Example 67: (Calc) Let R be the region in the first quadrant bounded by the graphs of $y = \cos\left(\frac{\pi x}{2}\right)$ and $y = x^2 - \frac{26}{5}x + 1$, shown in the figure.



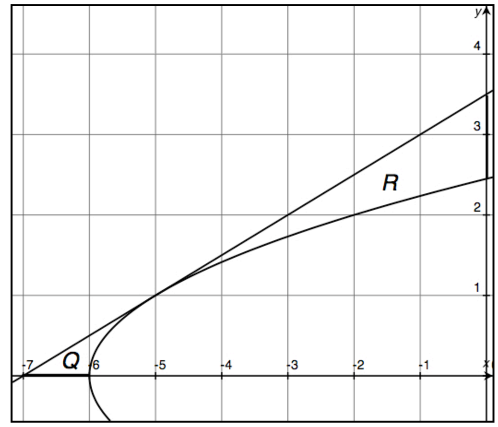
a) Find the area of R.

b) The vertical line $x = k$ splits the region R into two equal parts. Write, but do not solve, an equation involving integrals that solves for k.

c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

d) The region R models the surface of a piece of glass. At all points in R at a distance x from the y-axis, the thickness of the glass is given by $h(x) = 5 - x$. Find the volume of the glass.

Example 68: (Calc) Line l is tangent to $y^2 = x + 6$ at the point (-5, 1). Let R be the region bounded by $y^2 = x + 6$, line l, and the y-axis. Let Q be the region bounded by $y^2 = x + 6$, line l, and the x-axis. The figure to the right shows these regions.



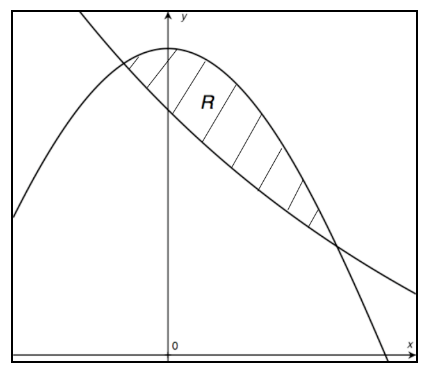
a) Show that the equation of line l is $y = \dfrac{x+7}{2}$.

b) Find the area of region R.

c) Find the volume of the solid generated by revolving region R about the line $y = 1$.

d) Find the area of region Q.

Example 69: (Calc) Let R be the region bounded by the graphs of $f(x) = 6\cos(2x) - 1$ and $g(x) = 4e^{-x} - x$.

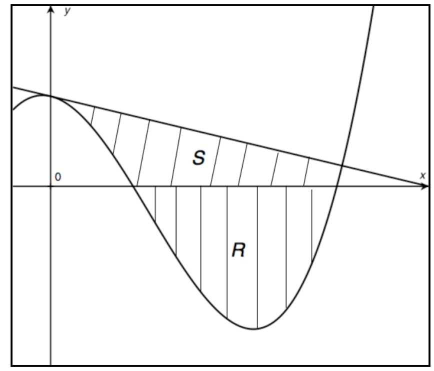


a) Find the area of R.

b) Find the volume of the solid generated when R is revolved about the x-axis.

c) The region R is the base of a solid with cross sections perpendicular to the x-axis as semi-circles with diameters extending from $y = f(x)$ to $y = g(x)$. Find the volume of the solid.

Example 70: (Calc) Let $f(x) = x^3 - 2x^2 - \dfrac{x}{3} + \cos x$ and line l be the line tangent to the graph of f at $x = 0$. Let R be the 4th quadrant region bounded by f and the x-axis and let S be the first quadrant area bounded by f, l, and the x-axis.

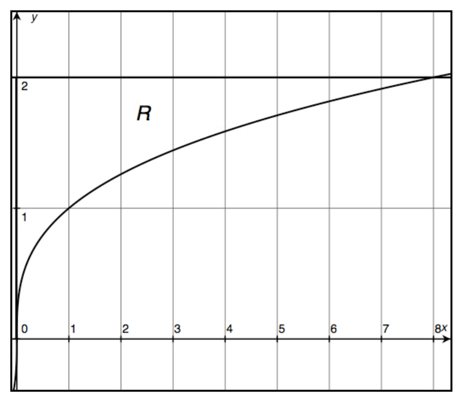


a) Find the area of region R.

b) Find the volume of the solid created by revolving region R about the x-axis.

c) Find the area of region S.

Example 71: Let R be the region bounded by the graph of $y = \sqrt[3]{x}$, the y-axis, and $y = 2$ as shown in the figure to the right.



a) Find the area of Region R.

b) Find the volume of the solid if region R is rotated about the y-axis.

c) Let the solid in part b) model a vase with x,y measured in inches. The vase fills with water at the rate of $\dfrac{6 \text{ in}^3}{\text{min}}$. When the water is 1-inch high, how fast does the water rise in the vase?

U. Derivative of an Inverse of a Function

What you are finding: There is probably no topic that confuses students (and teachers) more than inverses. The inverse of a function f is another function f^{-1} that "undoes" what f does. So $f^{-1}(f(x)) = x$. For instance, the inverse of adding 5 is subtracting 5. Start with any number x, add 5, then subtract 5, and you are back to x. Do not confuse the inverse f^{-1} with the reciprocal. $x^{-1} = \frac{1}{x}$ but $(f(x))^{-1} \neq \frac{1}{f(x)}$.

To find the inverse of a function, you replace x with y and y with x. The inverse to the function $y = 4x - 1$ is $x = 4y - 1$ or $y = \frac{x+1}{4}$.

In this section, you are concerned with finding the derivative of the inverse to a function: $\frac{d}{dx}\left[f(x)\right]^{-1}$

How to find it: The formula used is: $\frac{dy}{dx} = \frac{1}{f'(y)}$. But what I suggest, rather than memorizing this formula, is to switch x and y to find the inverse, and then take the derivative, using implicit differentiation:
$x = f(y) \Rightarrow 1 = f'(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)}$.

Example 72: Find the derivative of the inverse to $y = x^3$ at $x = 1$.

Example 73: Find the derivative of the inverse to $f(x) = 3x + \sin\pi x$ at $x = 6$.

Example 74: (Calc) Find the derivative of the inverse to $f(x) = x + 3\sin x$ at $x = 6$.

Example 75: If $f(x) = x^2$ (1^{st} quadrant), write the equation of the tangent line to $f^{-1}(x)$ at $x = 16$.

Example 76: If $f(x) = x^3 + x^2 + x + 1$, write the equation of the tangent line to $f^{-1}(x)$ at $x = 4$.

Example 77: In the chart to the right, selected values of x are given along with values of $f(x)$ and $f'(x)$.

x	$f(x)$	$f'(x)$
-2	3	-1
-1	2	-2
0	-1	4
1	1	-3

a) If f^{-1} is the inverse function of f, find the derivative of f^{-1} at $x = -1$: $\left(f^{-1}\right)'(-1)$.

b) Write an equation for the line tangent to the graph of $y = f^{-1}(x)$ at $x = 2$.

V. Differential Equations

What you are finding: A differential equation (DEQ) is in the form of $\frac{dy}{dx} = (\text{Algebraic Expression})$. The goal of solving a DEQ is to work backwards from the derivative to the function; that is to write an equation in the form of $y = f(x) + C$ (since the technique involves integration, the general solution will have a constant of integration). If the value of the function at some value of x is known (an initial condition), the value of C can be found. This is called a specific solution.

How to find it: In Calculus AB, the only types of DEQ's studied are called separable. Separable DEQ's are those than that can be in the form of $f(y)\ dy = g(x)\ dx$. Once in that form, both sides can be integrated with the constant of integration on only one side, usually the right.

Word problems involving change with respect to time are usually models of DEQ's. A favorite type is a problem using the words: The rate of change of y is proportional to some expression. The equation that describes this statement is: $\frac{dy}{dt} = k \cdot (\text{expression})$. The rate of change is usually with respect to time.
Usually there is a problem that requires you to create a slope field. Simply calculate the slopes using the given derivative formula and plot them on the given graph. Usually the slopes will be integer values.

Example 78: Consider the differential equation $\frac{dy}{dx} = (y+2)^2 \sin\left(\frac{x}{e}\right)$

a) There is a horizontal line with equation $y = k$ that satisfies the differential equation. Find the value of k.

b) There is a vertical line in the form of $x = w$, such that for any value of w, the function $y = f(x)$ has a horizontal tangent line. Find the smallest positive value of w.

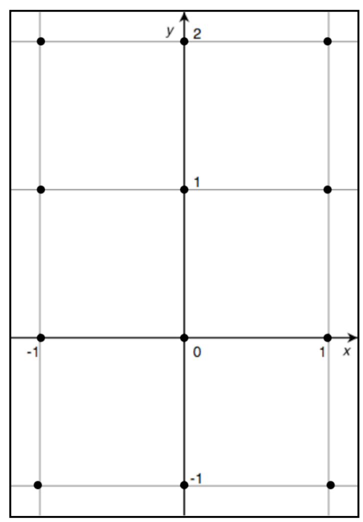
c) Find the particular solution $y = f(x)$ with the initial condition $f(0) = -1$.

Example 79: Consider the differential equation $\frac{dy}{dx} = \frac{-4x+2}{y}$.

a) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = -4$. Write an equation for the line tangent to the graph at (2, -4) and use it to approximate $f(1.8)$.

b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = -4$.

Example 80: Consider the differential equation $\frac{dy}{dx} = \frac{y+1}{x^3}, x \neq 0$.



a) On the axes provided, sketch a slope field for this DEQ.

b) Find the particular solution $y = f(x)$ to the differential equation given that $f(1) = 0$.

c) For the solution found in b), find $\lim_{x \to -\infty} f(x)$.

Example 81: (Calc) Let $R(t)$ represent the number of people in an office building who have heard a rumor at time *t* minutes where $t \geq 0$. $R(t)$ is increasing at a rate proportional to $160 - 2R(t)$, where the constant of proportionality is *k*.

a) If 10 people start the rumor, find $R(t)$ in terms of *t* and *k*.

b) If 40 people know the rumor 5 minutes after it started, find $R(t)$ in terms of *t*.

c) To the nearest person, how many people in the office have heard the rumor 10 minutes after it started?

d) Find $\lim_{t \to \infty} R(t)$.

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e) How many people have heard the rumor when the rumor is growing the fastest? Justify your answer.

Example 82: **(Calc)** At time $t = 0$ a biker is traveling at 900 ft/min when he decides not to do any pedaling. He slows down with a negative acceleration that is directly proportional to t^2. This brings the biker to a stop in 3 minutes.

a) Write an expression for the velocity of the biker at time t.

b) How far does the biker coast in that 3 minute time period?

Example 83: Consider the differential equation $xy' - y = 4x^2y, \quad x > 0$.

a) Find the general solution of the DEQ.

b) Find the particular solution of the DEQ passing through (1,1).