K. Function Analysis

What you are finding: You have a function $f(x)$. You want to find intervals where $f(x)$ is increasing and decreasing, concave up and concave down. You also want to find values of x where there is a relative minimum, a relative maximum, and points of inflection.
How to find them: Find critical values – values $x = c$ where $f'(c) = 0$ or $f'(c)$ is undefined.

$f(x)$ is increasing for values of k such that $f'(k) > 0$

$f(x)$ is decreasing for values of k such that $f'(k) < 0$.

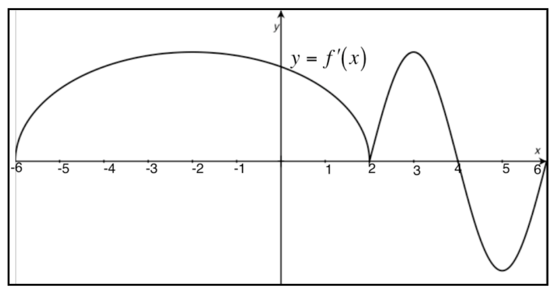
If $f(x)$ switches from increasing to decreasing at c, there is a relative maximum at $(c, f(c))$. If $f(x)$ switches from decreasing to increasing at c, there is a relative minimum at $(c, f(c))$. This is commonly called the first derivative test.

$f(x)$ is concave up for values of k such that $f''(k) > 0$.

$f(x)$ is concave down for values of k such that $f''(k) < 0$.

If $f(x)$ switches concavity at c, there is a point of inflection at $(c, f(c))$. If you want to find the actual maximum or minimum value or find the value of the function at specific points, you need to use accumulated area and the FTC.

Example 32: The figure to the right shows the graph of f', the derivative of the function f, for $-6 \le x \le 6$. The graph of f' has horizontal tangents at $x = -2$, $x = 3$, and $x = 5$ and is symmetric with respect to the x-axis from $x = 2$ to $x = 6$.



a) Find all values of x, for $-6 < x < 6$, at which f attains a relative maximum. Justify answer.

$x = 4$. f' changes from positive to negative at $x = 4$.

b) Find all values of x, for $-6 < x < 6$, at which f attains a relative minimum. Justify answer.

None. There is no value when f' changes from negative to positive.

c) Find all values of x, for $-6 < x < 6$, at which f has an inflection point. Justify answer.

$x = -2, 2, 3, 5$. $f'' > 0$ on $[-6,-2), (2,3), (5,6]$ $\quad f'' < 0$ on $(-2,2), (3,5)$
so f'' changes signs at $x = -2, 2, 3, 5$.

d) Find the value of x for which f has an absolute minimum. Justify answer.

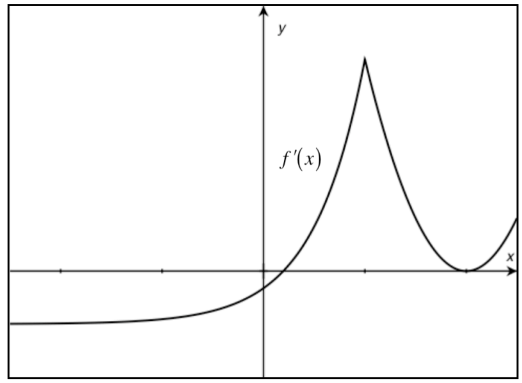
Since $f' > 0$ on $[-6,2)$, f is increasing on $[-6,2)$
Because of the symmetry, f increases then decreases the same amount on $(2,6]$
So the absolute minimum value of f occurs at $x = -6$. or

$$\int_{-6}^{2} f'(x) > 0, \int_{2}^{6} f'(x) = 0 \Rightarrow \int_{-6}^{6} f'(x) > 0$$

So the absolute minimum value of f occurs at $x = -6$.

Example 33: (Calc) The derivative of a function f is defined by

$$f'(x)=\begin{cases}5e^{2x-2}-1 & \text{for } -2.5\le x<1\\ 4(x-2)^2 & \text{for } 1\le x\le 2.5\end{cases}$$



The graph of the continuous function f' is shown in the figure on the right. $f(0)=-2$.

a) For what value of x does the graph of f have a point of inflection. Justify your answer.

$x=1, x=2$. At $x=1$, f' changes from increasing to decreasing.
At $x=2$, f' changes from decreasing to increasing.

b) For what values of x is f decreasing on [-2.5,2.5] ? Justify your answer.

$f'<0$ when $5e^{2x-2}-1<0 \Rightarrow\ 5e^{2x-2}<1 \Rightarrow e^{2x-2}<\dfrac{1}{5}$

$2x-2<\ln\left(\dfrac{1}{5}\right)\Rightarrow 2x-2<-\ln 5\Rightarrow x<\dfrac{2-\ln 5}{2}$

f is decreasing when $-2.5<x<\dfrac{2-\ln 5}{2}$

c) For what value of x does f attain an absolute minimum on [-2.5,2.5].

Since $f'<0$ for $-2.5\le x\le\dfrac{2-\ln 5}{2}$ and $f'>0$ for $\dfrac{2-\ln 5}{2}<x\le 2.5$
the absolute minimum for f occurs at $\dfrac{2-\ln 5}{2}$ or $x=.195$

d) What is the absolute minimum of f on [-2.5,2.5]?

$f(.195)-f(0)=\displaystyle\int_0^{.195}\left(5e^{2x-2}-1\right)dx \qquad f(.195)=f(0)+\int_0^{-.195}\left(5e^{2x-2}-1\right)dx$

$f(.195)=-2+(-.034)=-2.034$

e) Find $f(-2.5)$ and $f(2.5)$

$f(-2.5)-f(0)=\displaystyle\int_0^{-2.5}f'(x)\,dx \qquad f(-2.5)=f(0)+\int_0^{-2.5}\left(5e^{2x-2}-1\right)dx$

$f(-2.5)=-2+2.164=0.164$

$f(2.5)-f(0)=\displaystyle\int_0^{2.5}f'(x)\,dx \qquad f(2.5)=f(0)+\int_0^{1}\left(5e^{2x-2}-1\right)dx+\int_1^{2.5}4(x-2)^2\,dx$

$f(2.5)=-2+1.162+1.5=0.662$

L. Second Derivative Test

What you are finding: Students normally find points of relative maximum and relative minimum of $f(x)$ by using the first derivative test, finding points where a function switches from increasing to decreasing or vice versa. But students may be forced to use the 2nd derivative test, as seen below. **How to use it:** Find critical values c where $f'(c)=0$. a) If $f''(c)>0, f$ is concave up at c and there is a relative minimum at $x=c$. b) If $f''(c)<0, f$ is concave down at c and there is a relative maximum at $x=c$. c) If $f''(c)=0$, the 2nd derivative test is inconclusive. $\left(\text{ex.} f(x)=x^4 \text{ at } x=0\right)$

Example 34: Let f be the function defined by $f(x)=k\sqrt[3]{x}-e^x$ for $x \geq 0$, where k is a constant.

a) For what value of k does f have a critical point at $x = 1$?

$$f'(x)=\frac{k}{3x^{2/3}}-e^x \qquad f'(1)=\frac{k}{3}-e=0$$
$$k = 3e$$

b) For the value of k in part a), determine whether f has a relative maximum, relative minimum, or neither at $x = 1$. Justify your answer.

$$f''(x)=\frac{-2k}{9x^{5/3}}-e^x \qquad f''(1)=\frac{-2(3e)}{9}-e<0$$
Since $f'(1)=0$ and $f''(1)<0$, there is a relative maximum at $x=1$.

Example 35: Consider the differential equation $\frac{dy}{dx}=2x+\frac{y}{2}-4$.

a) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave down.

$$\frac{d^2y}{dx^2}=2+\frac{1}{2}\frac{dy}{dx}=2+\frac{1}{2}\left(2x+\frac{y}{2}-4\right)=2+x+\frac{y}{4}-2=x+\frac{y}{4}$$
$$\text{Concave down}: \frac{d^2y}{dx^2}<0 \Rightarrow x+\frac{y}{4}<0$$
f is concave down below the line $y<-4x$

b) Let $y=f(x)$ be a particular solution to the differential equation with the initial condition $f(0)=8$. Does f have a relative minimum, relative maximum or neither at $x = 0$? Justify.

$$\left.\frac{dy}{dx}\right|_{(0,8)}=2x+\frac{y}{2}-4=0+4-4=0 \quad \text{(Critical point)}$$
$$\left.\frac{d^2y}{dx^2}\right|_{(0,8)}=x+\frac{y}{4}=0+2=2>0. \qquad \text{Thus } f \text{ has a relative minimum at } (0,8)$$

M. Absolute Extrema

What you are finding: the highest value or lowest value of a function, typically in an interval. Another way of asking the question is to ask for the range of the function.

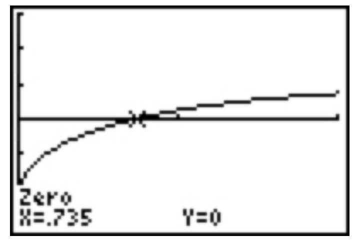
How to find it: a) find critical points (x-values where the derivative = 0 or is not defined).
b) use 1st derivative test to determine x-values when the function is increasing/decreasing.
c) evaluate function at critical points and at the endpoints. This may involve using the FTC to find accumulated change.

Example 36: (Calc) A particle moves along the y-axis such that its velocity v is given by $v(t) = 1 - 2\sin^{-1}(e^{-t})$ for $0 \le t \le 2$. Find the time when it reaches its lowest point.

To find extrema of position, we have to take the derivative of position which is the velocity that is given. So set $v(t) = 0$.

$$v(t) = 1 - 2\sin^{-1}(e^{-t}) = 0 \Rightarrow \sin^{-1}(e^{-t}) = \frac{1}{2}$$

$$e^{-t} = \sin\left(\frac{1}{2}\right) \Rightarrow t = -\ln\left[\sin\left(\frac{1}{2}\right)\right] = .735$$



Since, $v(t) < 0$ on $[0,.735)$ and $v(t) > 0$ on $(.735,2]$ the abs. min occurs at $t = .735$.

Example 37: Find the range of $f(x) = \dfrac{4x-4}{\sqrt{x^2-x+2}}$.

$$f'(x) = \frac{4\sqrt{x^2-x+2} - 4(x-1)\left(\frac{2x-1}{2\sqrt{x^2-x+2}}\right)}{x^2-x+2} = \frac{4(x^2-x+2) - 2(2x^2-3x+1)}{(x^2-x+2)^{3/2}} = \frac{2x+6}{(x^2-x+2)^{3/2}}$$

$$f'(x) = 0 \Rightarrow \frac{2(x+3)}{(x^2-x+2)^{3/2}} \Rightarrow x = -3$$

$f'(x) < 0$ if $x < -3$, $f'(x) > 0$ if $x > -3$ so there is a relative minimum at $x = -3$

$$f(-3) = \frac{-16}{\sqrt{14}}$$

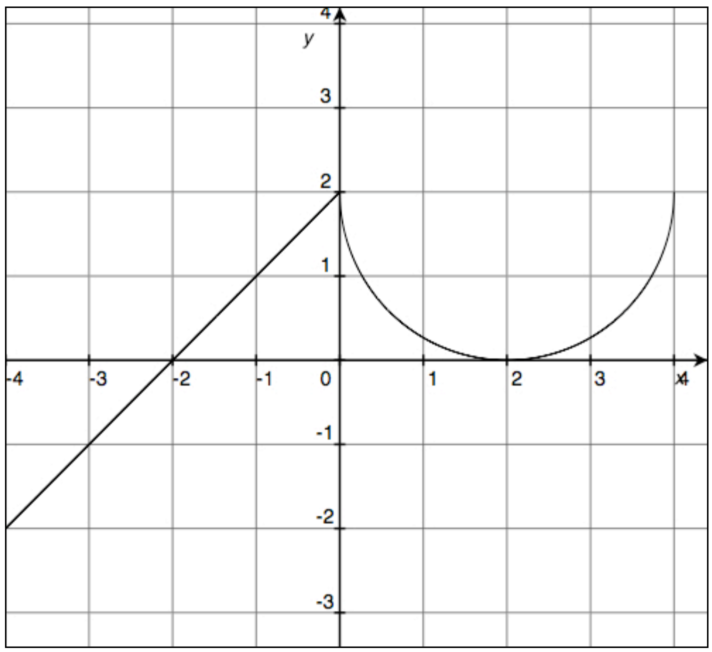
$f(x)$ is decreasing if $x < -3$, increasing if $x > -3$

$\lim_{x\to-\infty} f(x) = -4, \lim_{x\to\infty} f(x) = 4$ so range is $\left[\dfrac{-16}{\sqrt{14}}, 4\right)$

Note: this problem has a lot of algebraic manipulations. Later AP exams would probably have students show that $f'(x) = \dfrac{2x+6}{(x^2-x+2)^{3/2}}$ so they could find the range even if their algebra was faulty.

Example 38: The derivative of a function f is defined by $f'(x) = \begin{cases} x+2 & -4 \le x < 0 \\ g(x) & 0 \le x \le 4 \end{cases}$ where $g(x)$ is a semicircle of radius 2 and $f(0) = 8$.

For $-4 \le x \le 4$, find the minimum and maximum value of f. The graph of f' is below.



Since $f'(x) < 0$ for $x < -2$, f is decreasing on $[-4,-2)$
Since $f'(x) > 0$ for $x > -2$, f is increasing on $(-2,4]$
So there is a relative minimum at $x = -2$

$$\int_{-2}^{0} f'(x) = f(0) - f(-2) \Rightarrow f(-2) = f(0) - \int_{-2}^{0} f'(x) = 8 - 2 = 6$$

The absolute minimum value of f is 6.
The absolute maximum value could occur at $x = -4$ or $x = 4$.

$$x = -4 : \int_{-4}^{0} f'(x) = f(0) - f(-4) \Rightarrow f(-4) = f(0) - \int_{-4}^{0} f'(x) = 8 - 0 = 8$$

$$x = 4 : \int_{0}^{4} f'(x) = f(4) - f(0) \Rightarrow f(4) = f(0) + \int_{0}^{4} f'(x) = 8 + 8 - 2\pi = 16 - 2\pi$$

The absolute maximum value of f is $16 - 2\pi$

Note that this problem incorporates both the concept of accumulation as well as integration of the rate of change to give accumulated change.

Example 39: A 3-hour concert serves hot dogs as refreshments. The number of hot dogs that are cooked is modeled by a continuous function on the time interval $0 \le t \le 3$. In this model, rates are given as follows:

(i) the rate at which hot dogs are cooked is $f(t) = 10 - t^2$, measured in hundreds.
(ii) the rate at which hot dogs are sold is $g(t) = t^2 + t$, measured in hundreds.

When hot dogs are cooked, they are placed in warmers.
There are 500 hot dogs already in the warmers at the time the doors open ($t = 0$).

a) For $0 \le t \le 3$, at what time t is the number of hot dogs that are either on the grill or are in the warmers at a maximum? (hot dogs that have cooked or are cooking but not sold)

Since $f(t)$ and $g(t)$ are rates, they are derivatives. We are interested in where $f(t) - g(t) = 0$ (the rate of hot dogs being cooked equals the rate of hot dogs sold).

$10 - t^2 = t^2 + t$

$2t^2 + t - 10 = 0$

$(2t+5)(t-2) = 0 \Rightarrow t = 2.$

So the candidates for the absolute maximum are at $t = 0, t = 2, t = 3$

t (hours)	hot dogs cooking or cooked
0	500
2	$500 + 100\int_0^2 (10 - t^2 - t^2 - t)\,dt = 1767$
3	$500 + 100\int_0^3 (10 - t^2 - t^2 - t)\,dt = 1250$

At $t = 2$ hours, there will be a maximum number of hot dogs that either have been cooked or are in the warmers but not sold.

b) The concert goes 45 minutes longer than originally planned. No hot dogs are cooked during that time but the rate of hot dogs sold continues to be $g(t)$. Determine how many hot dogs will be left unsold.

At 3 hours, there are 1250 hot dogs left.

Hot dogs sold in between hour 3 and hour 3.375: $100\int_3^{3.75} (t^2 + t)\,dt = 1111$

$1250 - 1111 = 139$ hot dogs unsold.

N. Computation of Riemann Sums

What you are finding: Riemann sums are approximations for definite integrals, which we know represent areas under curves. There are numerous real-life models for areas under curves so this is an important concept. Typically these types of problems show up when we are given data points as opposed to algebraic functions.

How to find them: Given data points:

x	x_0	x_1	x_2	...	x_{n-2}	x_{n-1}	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	...	$f(x_{n-2})$	$f(x_{n-1})$	$f(x_n)$

Assuming equally spaced x-values: $x_{i+1} - x_i = b$

Left Riemann Sums: $S = b(x_0 + x_1 + x_2 + ... + x_{n-1})$

Right Riemann Sums: $S = b(x_1 + x_2 + ... + x_{n-1} + x_n)$

Trapezoids: $S = \frac{b}{2}(x_0 + 2x_1 + 2x_2 + ... + 2x_{n-2} + 2x_{n-1} + x_n)$

If bases are not the same (typical in AP questions), you have to compute the area of each trapezoid: $\frac{1}{2}(x_{i+1} - x_i)[f(x_i) + f(x_{i+1})]$

Midpoints: This is commonly misunderstood. For example, you cannot draw a rectangle halfway between $x = 2$ and $x = 3$ because you may not know $f(2.5)$. You can't make up data. So in the table above the first midpoint rectangle would be drawn halfway between x_0 and x_2 which is x_1. So assuming that the x-values are equally spaced, the midpoint sum would be $S = 2b(x_1 + x_3 + x_5 + ...)$.

Example 40: (Calc) James is doing some biking. His velocity in feet/minute is given by a differentiable function V of time t. The table to the right shows the velocity measured every 3 minutes for a 30-minute period.

t (minutes)	$v(t)$ ft per minute
30	880
33	1,056
36	1,408
39	440
42	616
45	836
48	264
51	550
54	610
57	1,200
60	620

a) Find the difference in the estimation of $\int_{30}^{60} V(t)\, dt$ using left Riemann sums and right Riemann sums.

$$\text{Left}: \int_{30}^{60} V(t)\, dt \approx 3(880 + 1056 + 1408 + ... + 1200)$$

$$\text{Right}: \int_{30}^{60} V(t)\, dt \approx 3(1056 + 1408 + ... + 1200 + 620)$$

These calculations have the same numbers except for the first and last so the difference $= 3(880 - 620) = 780$

b) Use a midpoint Riemann sum with 5 subdivisions to approximate $\frac{1}{5280}\int_{30}^{60} V(t)\, dt$. Using correct units, explain the meaning of your answer.

$$\frac{1}{5280}\int_{30}^{60} V(t)\, dt \approx \frac{6(1056 + 440 + 836 + 550 + 1200)}{5280} = 4.639$$

James travels approximately 4.639 miles between minute 30 and minute 60.

Example 41: A weather balloon travels along a straight line vertically. During the time interval $0 \le t \le 60$ seconds, the balloon's velocity v, measured in meters/sec, and acceleration a, measured in meters per second per second, are continuous functions. The table below shows selected values of these functions.

t (sec)	0	5	15	30	40	55	60
$v(t)$ m/sec	-30	-40	-20	-5	0	20	10
$a(t)$ m/sec^2	2	4	1	1	2	5	3

a) Approximate $\int_0^{60} |v(t)|\,dt$ using a right Riemann sum and explain its meaning in terms of the balloon's motion using appropriate units.

$$\int_0^{60} |v(t)|\,dt \approx 5(40) + 10(20) + 15(5) + 10(0) + 15(20) + 5(10) = 825$$

The balloon travels approximately 825 meters.

b) Approximate $\int_0^{60} a(t)\,dt$ using a trapezoidal approximation with three subintervals and explain its meaning in terms of the balloon's motion using appropriate units.

$$\int_0^{60} a(t)\,dt \approx \frac{15}{2}(2+1) + \frac{25}{2}(1+2) + \frac{20}{2}(2+3) = 110$$

The balloon's change in velocity from $t = 0$ to $t = 60$ is approximately 110 m/sec.

c) Find the exact value of $\int_0^{60} a(t)\,dt$.

$$\int_0^{60} a(t)\,dt = \int_0^{60} v'(t)\,dt = v(60) - v(0) = 10 - (-30) = 40 \text{ m/sec.}$$

Example 42: (Calc) Let $F(x) = \int_1^{x} \sin(\ln x)\,dx$. Use the trapezoidal rule with four equal subdivisions to approximate $F(2)$.

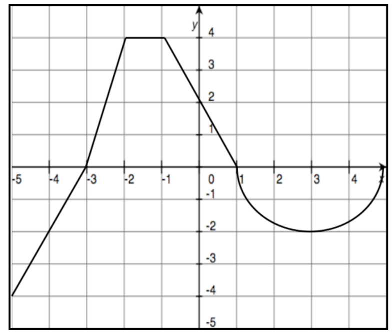
$$F(2) \approx \frac{1}{2}\left(\frac{1}{4}\right)\left[\sin(\ln(1)) + 2\sin(\ln(1.25)) + 2\sin(\ln(1.5)) + 2\sin(\ln(1.75)) + \sin(\ln(2))\right]$$

$$F(2) \approx \frac{1}{8}(1.932) = 0.367$$

O. Accumulation Function / Fundamental Theorem of Calculus (FTC)

What you are finding: the accumulation function looks like this: $F(x)=\int_0^x f(t)\,dt$. It represents the accumulated area under the curve *f* starting at zero (or some value) and going out to the value of *x*. The variable *t* is a dummy variable. It is important to believe that this is a function of x.
How to find it: Typically this is used with particle motion. When *f* represents a velocity, the accumulation function will represent the displacement of the particle from time = 0 to time = *x* (section S in this manual). This type of function goes hand-in-hand with absolute max/min problems.

Example 43: A graph of *f* is made up of lines and a semi-circle as shown to the right. Let $F(x)=\int_{-1}^{x} f(t)\;dt$. Find $F(5)$ and $F(-5)$.



$$F(5)=\int_{-1}^{5} f(t)\,dt=\frac{1}{2}(2)(4)-2\pi=4-2\pi$$

$$F(-5)=\int_{-1}^{-5} f(t)\,dt=-\int_{-5}^{-1} f(t)\,dt=-\left(-\frac{1}{2}(2)(4)+\frac{1}{2}(1)(4)+4\right)=-2$$

Example 44: The table below gives values of the continuous velocity function of an elevator in a building at selected times. The elevator is 40 feet high at time $t=0$.

t (seconds)	0	20	30	50	70	100	120	130	160
$v(t)$ (ft/sec)	6	8	-2	4	0	-5	-1	2	3

a. Use a right Riemann sum with 4 intervals to approximate $40+\int_0^{120} v(t)\,dt$ and explain its meaning. Show your calculations.

$$40+\int_0^{120} v(t)\;dt\approx 40+30(-2)+40(0)+50(-1)+40(3)=50$$

The elevator is approximately 50 feet high after 120 seconds.

b. Use a trapezoidal approximation with 5 trapezoids to approximate $\int_{20}^{120}|v(t)|\;dt$ and explain its meaning. Show your calculations.

$$\int_{20}^{120}|v(t)|\;dt\approx\frac{10}{2}(8+2)+\frac{20}{2}(2+4)+\frac{20}{2}(4+0)+\frac{30}{2}(0+5)+\frac{20}{2}(5+1)=285$$

The elevator traveled approximately 285 feet between $t=20$ and $t=120$

c. Using the table above, what is the minimum number of values of *t* such that $\frac{d}{dt}\int_0^t v(k)\,dk=0$? Justify your answer.

$$\frac{d}{dt}\int_0^t v(k)\,dk=\frac{d}{dt}\big[y(t)-y(0)\big]\,dt=v(t).\ \left(\text{Easier: The 2nd FTC states that } \frac{d}{dt}\int_0^t v(k)\,dk=v(t)\right)$$

There is one given value of *t* for which $v(t)=0$ but by the IVT, $v(t)$ must equal 0 between $t=20$ and $t=30$, $t=30$ and $t=50$, and $t=120$ and $t=130$. The answer is 4.

This type of problem is fair game to show up on an exam. It illustrates the meaning of the Fundamental Theorem of Calculus.

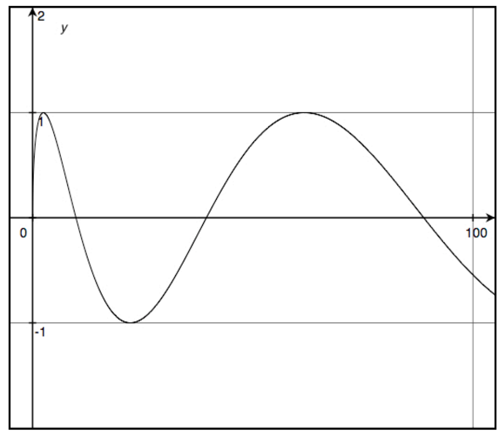
Example 45: (Calc) $F(x)=\int 2^{x^2-2x+1}dx$ and $F(1)=2$, find $F(2)$.

Many students give up on this type of problem because they cannot integrate 2^{x^2-2x+1}.

But realize that $F(2)-F(1)=\int_1^2 2^{x^2-2x+1}dx$ and thus $F(2)=F(1)+\int_1^2 2^{x^2-2x+1}dx$

Using the calculator : $F(2)=2+\int_1^2 2^{x^2-2x+1}dx=2+1.288=3.288$

Example 46: A particle moves along the x-axis such that its velocity v at time $t\ge 0$ is given by $v(t)=\sin\sqrt{t}$ for $0\le t\le 100$. The graph of $v(t)$ is shown to the right. The position of the particle at time t is $x(t)$ and its position at time $t=0$ is $x(0)=10$. Find the times when the particle is farthest to the right and farthest to the left. Explain your answer.



$v(t)=0\Rightarrow\sin\sqrt{t}=0 \qquad \sqrt{t}=0,\pi,2\pi,3\pi$ so $t=0,\pi^2,4\pi^2,9\pi^2$

$v(t)>0$ if $0<t<\pi^2$ and $4\pi^2<t<9\pi^2$ so particle is farthest to the right at $t=\pi^2$ or $t=9\pi^2$

$x(\pi^2)=10+\int_0^{\pi^2}\sin\sqrt{t}\,dt,\ x(9\pi^2)=10+\int_0^{9\pi^2}\sin\sqrt{t}\,dt$

$x(9\pi^2)>x(\pi^2)$ as the displacement between $t=\pi^2$ and $t=9\pi^2$ is positive

So the particle is farthest to the right at $t=9\pi^2$

$v(t)<0$ if $\pi^2<t<4\pi^2$ and $9\pi^2<t<100$

The particle is farthest to the left at either $t=0$, $t=4\pi^2$ or $t=100$

$x(0)=10,\ x(4\pi^2)=10+\int_0^{4\pi^2}\sin\sqrt{t}\,dt, x(100)=10+\int_0^{100}\sin\sqrt{t}\,dt$

Displacement between $t=0$ and $t=4\pi^2$ is negative and between $t=4\pi^2$ and $t=100$ is positive

So the particle is farthest to the left at $t=4\pi^2$.

Note that a graphing calculator is not necessary to determine t - values where the particle is farthest or the left and right. If the question asked to find how far to the left or right the particle had traveled, then a calculator would be necessary to calculate the definite integrals.

P. Interpretation of a Derivative as a Rate of Change

What you are finding: As mentioned in section I (Related Rates), a quantity that is given as a rate of change needs to be interpreted as a derivative of some function. Typical problems ask for the value of the function at a given time. These problems can be handled several ways:

a) solving a Differential Equation with initial condition (although DEQ's may not have even been formally mentioned yet)

b) Integral of the rate of change to give accumulated change. This uses the fact that:

$$\int_a^b R'(t)\,dt = R(b) - R(a) \text{ or } R(b) = R(a) + \int_a^b R'(t)\,dt$$

Example 47: A car's gas tank contains 4 gallons. A gas pump can fill the tank at the rate of $\sqrt{9-t}$ gallons per minute for $0 \le t \le 10$ minutes. How many gallons of gas are in the tank at $t = 5$ minutes?

Method 1: $G(t)$ = gallons put in the tank in the first t minutes

$$\frac{dG}{dt} = (9-t)^{\frac{1}{2}} \Rightarrow G(t) = -\frac{2}{3}(9-t)^{\frac{3}{2}} + C$$

$$G(0) = 4 \Rightarrow G(0) = -\frac{2}{3}(9)^{\frac{3}{2}} + C = 4 \Rightarrow C = 22$$

$$G(t) = 22 - \frac{2}{3}(9-t)^{\frac{3}{2}} \Rightarrow G(5) = 22 - \frac{2}{3}(9-5)^{\frac{3}{2}} = 16.667 \text{ gallons}$$

Method 2: $\int_0^5 G'(t)\,dt = G(5) - G(0) \Rightarrow G(5) = G(0) + \int_0^5 G'(t)\,dt$

$$G(5) = 4 + \int_0^5 G'(t)\,dt = \int_0^5 (9-t)^{\frac{1}{2}}\,dt \text{ (If calculator is allowed, this is easier)}$$

$$G(5) = 4 + \left[-\frac{2}{3}(9-t)^{\frac{3}{2}}\right]_0^5 = 4 - \frac{16}{3} + 18 = 16.667 \text{ gallons}$$

Example 48: (Calc) Frankenstein Electronics makes gold-plated HDMI cables for HD TV's that sell for \$9 a foot. Frankenstein says that the cost of creating an x-foot cable is $\frac{\sqrt{x^3}}{3}$ dollars.

a) Write an expression involving an integral that represents Frankenstein's profit on a cable of length k.

$$\text{Profit} = 9k - \int_0^k \frac{\sqrt{x^3}}{3}\,dx$$

b) Find the maximum profit that Frankenstein could earn on a cable. Justify your answer.

$$P'(k) = 9 - \frac{\sqrt{k^3}}{3} = 0 \Rightarrow 27 = k^{\frac{3}{2}} \Rightarrow k = 9.$$

P' changes from positive to negative at $k = 9$ so there is a rel. maximum there.

$$P(9) = 81 - \int_0^9 \frac{\sqrt{x^3}}{3}dx = 81 - 32.4 = \$48.60$$

Example 49: (Calc) The Washington DC subway system uses cards for people to enter the system and for people to leave. So it is known how many people enter and leave the system at any one time. In a certain 12-hour period, people enter the system at a rate modeled by the function *E.*

$$E(t) = 2 + 3\sin\left(\frac{2\pi t}{25}\right)$$

In the same 12-hour period, people leave the system at a rate modeled by the function *L.s*

$$L(t) = -0.11t^2 + 1.41t + 0.57$$

Both $E(t)$ and $L(t)$ are measured in thousands of people per hour and t is measured in hours for $0 \le t \le 12$. At $t = 0$, there are 5,000 people on the subway system.

a) How many more people enter the subway during the second 6 hours than the first 6 hours?

$$\boxed{1000\left(\int_6^{12} E(t)\,dt - \int_0^6 E(t)\,dt\right) \approx 1{,}405 \text{ people}}$$

b) Write an expression for $P(t)$, the total number of people on the subway at time t.

$$\boxed{P(t) = 5000 + 1000\int_0^t \left(E(x) - L(x)\right) dx}$$

c) For what time is the number of people on the subway system a maximum? What is the maximum value? Justify your answers.

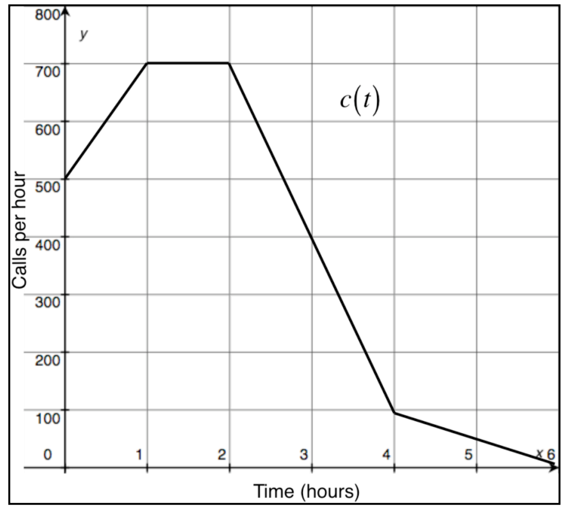
$$P'(t) = \frac{d}{dt}\left(5000 + \int_0^t \left(E(x) - L(x)\right) dx\right) = E(t) - L(t) \quad \text{(2nd FTC)}$$

$P'(t) = 0 \Rightarrow E(t) - L(t) = 0 \Rightarrow$ Critical values $t = 4.793,\ 9.370$

t	0	4.793	9.370	12
$P(t)$	5000	7358	7033	7779

The number of people on the subway system is at a maximum when $t = 12$ hrs.
There are 7,779 people on the subway at that time.

Example 50: (Calc) A new Broadway show is opening and people call in for tickets. There are 300 people waiting on hold on the phone when the operators start processing calls. They can handle 400 calls per hour. The graph to the right consisting of four straight lines shows the rate $c(t)$ at which new calls are coming in. Time t is measured in hours from when the processing starts. Assume that no one hangs up when on hold.



a) How many people have called in by the end of the 6-hour period?

$$300 + \int_0^6 c(t)\,dt$$

$$300 + \left(\frac{1}{2}\right)(500 + 700) + 700 + 2\left(\frac{1}{2}\right)(700 + 100) + \frac{1}{2}(2)(100) = 2500 \text{ people}$$

b) Is the number of people who are waiting on hold increasing, decreasing, or staying the same between hour 1 and hour 2? Give a reason for your answer.

it is increasing because calls are being processed at 400 an hour and for $1 < t < 2,\ c(t) > 400$.

c) At what time t is the number of people on hold the longest? How many people are on hold at that time?

$$H(t) = 300 + \int_0^t c(x)\,dx - 400t$$

$$H'(t) = c(t) - 400 = 0$$

$c(t) > 400$ for $t < 3, c(t) < 400$ for $t > 3$

So the number of people on hold is greatest at 3 hours

There will be $300 + \int_0^3 c(t)\,dt - 400(3) = 300 + 1850 - 1200 = 950$ people on hold

d) Write but do not solve an equation involving an integral expression of c whose solution gives the earliest time k when there is no one waiting on hold.

$$H(k) = 300 + \int_0^k c(t)\,dt - 400k = 0$$

Q. Derivative of Accumulation Function (2^{nd} FTC)

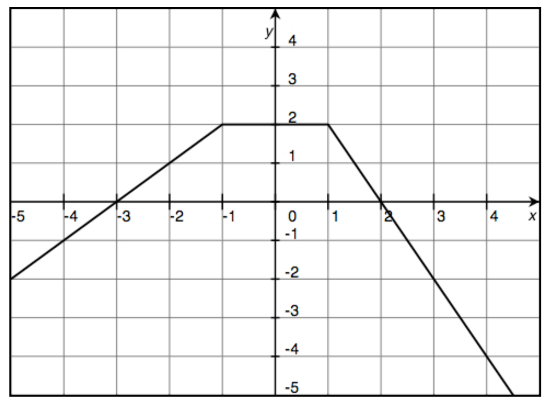
What you are finding: You are looking at problems in the form of $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt$. This is asking for the rate of change with respect to *x* of the accumulation function starting at some constant (which is irrelevant) and ending at that variable *x*. It is important to understand that this expression is a function of *x*, not the variable *t*. In fact, the variable *t* in this expression could be any variable (except *x*).

How to find it: You are using the 2^{nd} Fundamental Theorem of Calculus that says: $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$.

Occasionally you may have to use the chain rule that says $\dfrac{d}{dx}\displaystyle\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$.

Example 51: Let $f(x)$ be defined by the graph to the right whose domain is [-5,5].

Let $F(x) = \displaystyle\int_{-3}^x f(t)\,dt$ and $F(-3) = 0$



a) Put $F(4)$, $F'(4)$ and $F''(4)$ in order from largest to smallest.

$$F(4) = \int_{-3}^4 f(t)\,dt = \frac{1}{2}(2)(2) + (2)(2) + 1 - \frac{1}{2}(2)(4) = 3$$
$$F'(x) = \frac{d}{dx}\int_{-3}^x f(t)\,dt = f(x) \Rightarrow F'(4) = f(4) = -4$$
$$F''(x) = f'(x) = -2 \Rightarrow F''(4) = -2$$
$$\text{Largest}: F(4) \quad \text{Middle}: F''(4) \quad \text{Smallest}: F'(4)$$

b) Find the equation of the tangent line to *F* at *x* = 4.

$$y - y_1 = m(x - x_1) \qquad y - 3 = -4(x-4) \Rightarrow y = 19 - 4x$$

c) Use the results of b) to approximate the value of *F* at *x* = 4.1. Does this value over-approximate or under-approximate $F(4.1)$? Justify your answer.

$F(4.1) \approx 19 - 4(4.1) = 2.6$.
Since the slope is negative and F is concave down, 2.6 is an over-approximation.

d) Find the value of *x* where *F* has a maximum. Justify your answer.

$F'(x) = f(x) = 0 \qquad x = -3, x = 2$
$F'(x) < 0$ on $(-5,-3)$ and $(2,5)$ and $F'(x) > 0$ on $(-3,2)$
So f has a maximum at $x = 2$.

Example 52: Let $f(x)$ be a function that is continuous on the interval $[-1,3)$. The function is twice-differentiable except at $x = 1$. The function f and its 1st and 2nd derivatives have the properties in the table below, where DNE means that the derivative doesn't exist.

x	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	3	Positive	0	Negative	-5	Negative	0	Negative
$f'(x)$	-10	Negative	0	Negative	DNE	Positive	0	Negative
$f''(x)$	6	Positive	0	Negative	DNE	Negative	0	Negative

a) Let g be the function $g(x) = \int_2^x f(t)\,dt$ on the interval $(-1,3)$. For $-1 < x < 3$, find all values of x at which g has a potential relative extremum. Classify each as a relative maximum, relative minimum, or neither. Justify your answer.

$g'(x) = \frac{d}{dx}\int_2^x f(t)\,dt = f(x)$ (by the 2nd FTC).

$g'(x) = f(x) = 0$ at $x = 0, x = 2$

At $x = 0$, g' changes from positive to negative so g has a relative maximum at $x = 0$.

At $x = 2$, the sign of g' does not change so there is no relative extremum at $x = 2$.

b) For the function g defined in part a), find all values of x, for $-1 < x < 3$, at which the graph of g has a point of inflection. Justify your answer.

$g'(x) = f(x)$ so $g''(x) = f'(x)$

g has an inflection point at $x = 1, x = 3$ because $g'' = f'$ changes signs at those values.
Note the fact that f' is not defined at $x = 1$ doesn't change this fact. Also note that the values of $f''(x)$ are irrelevant in this problem.

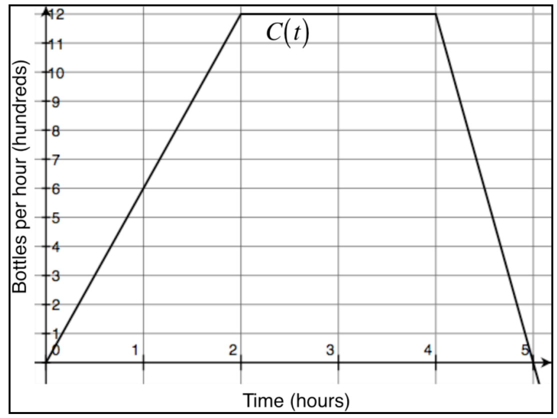
Example 53: The functions f and g are differentiable for all real numbers and g is strictly decreasing. The table below gives values of the functions and their first derivatives at selected values of x. Let h be the function given by $h(x) = \int_{-1}^{g(x)} f(t)\,dt$. Find the value of $h'(-1)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	3	5	4	-3
-1	5	1	0	-2
0	8	-3	-2	-1
1	-2	-4	-3	-2

By the 2nd FTC and the chain rule, $h'(x) = f(g(x)) \cdot g'(x)$

$h'(-1) = f(g(-1)) \cdot g'(-1) = f(0) \cdot (-2) = 8(-2) = -16$

Example 54: A juice bottling company has new bottles of juice emerge from a sanitizer and placed on a conveyor belt. From the conveyor belt, they are placed into boxes for shipping.



Let $C(t)$, made up of three straight lines shown by the graph on the right, represent the rate, measured in hundreds of bottles per hour that bottles of juice go onto the conveyor belt.

The rate that bottles are boxed is $B(t) = 900$, measured in bottles/hr. At the start of the 5-hour shift, there are 800 bottles on the belt. Suppose $F(t) = \int_0^t \left[C(x) - B(x)\right] dt$.

a) Write and calculate an expression using F that represents the number of bottles on the belt at 2 hours.

$$800 + F(2) = 800 + \int_0^2 \left[C(x) - B(x)\right] dx = 800 + 1200 - 1800 = 200$$

b) Find $F'(1)$ and express its meaning.

$$F'(t) = \frac{d}{dt}\int_0^t \left[C(x) - B(x)\right] dx = C(t) - B(t) \qquad F'(1) = C(1) - B(1) = 600 - 900 = -300$$

The number of bottles on the belt is decreasing at 300 bottles per hour at $t = 1$ hour.

c) When will the number of bottles on the conveyor belt be at a minimum and how many bottles will be on the belt at that time? Justify your answer.

$F'(t) = C(t) - B(t) = 0 \Rightarrow C(t) = 900$ at $t = 1.5,\ t = 4.25$

$F'(t) < 0$ between $t = 0$ and $t = 1.5$, $t = 4.25$ and $t = 5$

$F'(t) > 0$ between $t = 1.5$ and $t = 4.25$

so the values when bottles can be at a minimum are at $t = 1.5, t = 5$

At $t = 1.5$, the bottles on the belt are $800 + 675 - 1350 = 125$

At $t = 5$, the bottles on the belt are $800 + 1200 + 2400 + 600 - 4500 = 500$

The minimum number of bottles on the belt is 125 at $t = 1.5$.

d) Assume that after the 5-hour shift ends, another 5-hour shift starts up with the same graph of $C(t)$ and the same boxing rate of $B(t)$. Write, but do not solve, an equation in terms of time $t = k$ for when the conveyor belt is empty.

$$800 + \int_0^k \left[C(x) - B(x)\right] dx = 0 \quad \text{or} \quad 800 + 500 + \int_5^k \left[C(x) - B(x)\right] dx = 0$$