A. Horizontal Asymptotes

What you are finding: Typically, a horizontal asymptote (H.A.) problem is asking you find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

How to find it: Express the function as a fraction. If both numerator and denominator are polynomials,

a) If the higher power of x is in the denominator, the H.A. is $y = 0$.
b) If both the numerator and denominator have the same highest power, the H.A. is the ratio of the coefficients of the highest power term in the numerator and the coefficient of the highest power term in the denominator.
c) If the higher power of x is in the numerator, there is no H.A.

Note that this type of problem hasn't been asked in a free response question since 1995. It is taught in precalculus but teachers incorporate the concepts of limits in calculus to review the concept.

Note: L'Hopital's rule can be used to find these limits if students have learned it. L'Hopital's rule is not in the AB curriculum. I recommend teaching it at the end of the year if there is time.

Example 1: Given $f(x) = \dfrac{6x+1}{\sqrt{4x^2+6x+9}}$, write an equation for any horizontal asymptotes of $f(x)$.

This is an example of a function that has the highest power of x in both numerator and denominator. $\left(\sqrt{4x^2} = 2x\right)$

$\lim_{x\to\infty} f(x) = \dfrac{6}{2} = 3$

$\lim_{x\to-\infty} f(x) = \dfrac{-6}{2} = -3$ (the numerator will be negative, the denominator is positive)

So the asymptotes are $y = 3, y = -3$

Example 2: Show that $f(x) = \dfrac{\sin x}{e^x}$ has a horizontal asymptote on one side of the y-axis but not on the other side.

$\lim_{x\to\infty} \dfrac{\sin x}{e^x} = 0$ (numerator will always be between -1 and 1 but denominator will get bigger without bound)

$\lim_{x\to-\infty} \dfrac{\sin x}{e^x}$ (numerator will always be between -1 and 1 but denominator will get smaller without bound)

So there is a horizontal asymptote to the right of the y-axis at $y = 0$ (the x-axis)

B. Approximate Rate of Change

What you are finding: An approximation of the derivative of a function. Typically this type of question occurs when you have a table of values and not a specific function.

How to find it: If you are given a table of n values.

x	x_1	x_2	...	x_i	...	x_{n-1}	x_n
y	y_1	y_2	...	y_i	...	y_{n-1}	y_n

To find the approximate rate of change at $x = i$, you can either use:

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} \text{ or } \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \text{ or } \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}}$$

Graphically, you are finding the slope of the secant line between two points.

Example 3: a) While cruising Glacier Bay in Alaska, Lisa took a lot of pictures. The total number of pictures she took is given by a differentiable function L of time t. A table of selected values of L for the time interval $0 \le t \le 2.5$ is below. Use data from the table to approximate $L'(1.7)$. Show the computation that leads to your answer, explain the meaning of your answer and specify units of measure.

t (hours)	0.0	0.5	1.0	1.5	2.0	2.5
$L(t)$ (pictures)	0	20	45	75	90	100

$$L'(1.7) \approx \frac{90-75}{2-1.5} = 30 \text{ pictures/hr}$$

At $t = 1.7$ hours, Lisa's rate of picture taking is approximately 30 pictures/hour.

b) Ted also took a lot of pictures. His rate of picture taking in pictures per hour during the cruise is given by a differentiable function T of time t. A table of selected values of T for the time interval $0 \le t \le 2.5$ is below. Use data from the table to approximate $T'(1.7)$. Show the computation that leads to your answer, explain the meaning of your answer and specify units of measure.

t (hours)	0.0	0.5	1.0	1.5	2.0	2.5
$T(t)$ (pictures per hour)	40	28	53	55	45	24

$$T'(1.7) \approx \frac{45-55}{2-1.5} = -20 \text{ pictures/hr}^2$$

At $t = 1.7$ hours, Ted's rate of picture taking is approximately decreasing by 20 pictures per hour per hour.

Note the subtle differences between a) and b). In part a), you are given the function that represents the <u>number</u> of pictures that have been taken. In part b) you are given the <u>rate</u> of pictures taken. In part b), the function was described as $T(t)$ but since it is a rate, it could just as easily be described as $T'(t)$. That doesn't change the units in your answer. In part b), we can also use the 2nd FTC/ accumulation function to estimate the number of pictures Ted took (Section O).

Example 4: The velocity of a particle moving along the y-axis is modeled by a differentiable function v, where the position y is measured in feet and time t is measured in seconds. Selected values of $v(t)$ and $a(t)$ - the acceleration of the particle are given in the table below.

t	0	10	18	30	35	50
$v(t)$	4	12	9	-20	-6	2
$a(t)$	2	3	-1	Unknown	-3.5	3

a) Approximate the acceleration at $t = 15$ sec. Show your computation and indicate units.

$$a(15) \approx \frac{9-12}{18-10} = -\frac{3}{8}\frac{\text{ft}}{\text{sec}^2}$$

b) Approximate the acceleration at $t = 30$ sec. Show your computation and indicate units.

$$a(30) \approx \frac{-20-9}{30-18} = -\frac{-29}{12}\frac{\text{ft}}{\text{sec}^2} \quad \text{or} \quad a(30) \approx \frac{-6+20}{35-30} = \frac{14}{5}\frac{\text{ft}}{\text{sec}^2} \quad \text{or} \quad a(30) \approx \frac{-6-9}{35-18} = \frac{-15}{17}\frac{\text{ft}}{\text{sec}^2}$$

Students can get confused with the words "average rate of change." This means the average value of the rate of change. Typically, this involves finding the average value of a function, which involves the 2nd Fundamental Theorem of Calculus. Section R in this manual stresses the difference between average value as opposed to average rate of change, but here is a problem that illustrates this concept.

Example 5 (Calc): Let $F(x) = \int_0^x \cos(t^2 + t + 1)\,dt$.

a) Find the average rate of change of F on [3,7].

Since we have the function, we can actually find the average rate of change and not just and estimation. Our formula still holds though, we are finding: $\dfrac{F(7) - F(3)}{7-3}$

This gives: $$\frac{\int_0^7 \cos(t^2 + t + 1)\,dt - \int_0^3 \cos(t^2 + t + 1)\,dt}{7-3} = \frac{\int_3^7 \cos(t^2 + t + 1)\,dt}{4} = -.007$$

If this was attacked as an average value of a function problem, we would be finding the average value of $\cos(t^2 + t + 1)$ on $[3,7]$.

The formula that is used to find the average value of a function F on $[a,b]$ is given by:

$$F_{\text{avg}} = \frac{\int_a^b F(x)}{b-a} = \frac{\int_3^7 \cos(t^2 + t + 1)\,dt}{4} = -.007.$$

b) Find the average rate of change of $F'(x)$ on [3,7].

This uses the 2nd Fundamental Theorem (2nd FTC) which is covered in section Q.

$$F'(x) = \frac{d}{dx}\int_0^x \cos(t^2 + t + 1)\,dt = \cos(x^2 + x + 1)$$

Avg. rate of change of $F'(x) = \dfrac{F'(7) - F'(3)}{7-3} = \dfrac{\cos(57) - \cos(13)}{4} = .204$

C. Intermediate Value Theorem (IVT)

What it says: If you have a continuous function on $[a,b]$ and $f(b) \neq f(a)$, the function must take on every value between $f(a)$ and $f(b)$ at some point between $x = a$ and $x = b$. For instance, if you are on a road traveling at 40 mph and a minute later you are traveling at 50 mph, at some time within that minute, you must have been traveling at 41 mph, 42 mph, and every possible value between 40 mph and 50 mph.

Example 6: A truck travels along a straight track. During the time interval $0 \le t \le 30$ seconds, the truck's velocity in meters per second is a differentiable function. The table below shows selected values of this function.

t (sec)	0	5	8	12	20	24	30
$v(t)$ (m/sec)	-10	-25	-5	2	8	-1	3

For $0 \le t \le 30$, what is the minimum number of times that the truck must have been stopped? Justify your answer.

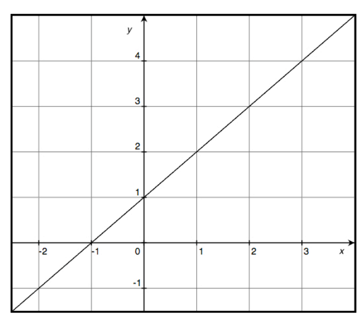
There are at least 3 times that the truck must have been stopped.
Since the velocity function is differentiable, it is thus continuous.
Since $v(8) = -5 < 0 < 2 = v(12)$, the IVT guarantees a t in $(8,12)$ so that $v(t) = 0$
Since $v(20) = 8 > 0 > -1 = v(24)$, the IVT guarantees a t in $(20,24)$ so that $v(t) = 0$
Since $v(24) = -1 < 0 < 3 = v(30)$, the IVT guarantees a t in $(24,30)$ so that $v(t) = 0$

Example 7: The functions f and g are continuous. The function h is given by $h(x) = f(g(x)) - x$. The table below gives values of the functions. Explain why there must be a value t for $1 < t < 4$ such that $h(t) = -1$.

x	1	2	3	4
$f(x)$	0	8	-3	6
$g(x)$	3	7	-1	2

$h(1) = f(g(1)) - 1 = f(3) - 1 = -3 - 1 = -4 \qquad h(4) = f(g(4)) - 4 = f(2) - 4 = 8 - 4 = 4$
Since $h(1) < -1 < h(4)$, by the IVT, there exists a value t in $(1,4)$ such that $h(t) = -1$.

Example 8: The straight-line function f is shown by the graph to the right. Define $F(x) = \int_{-2}^{x} f(t)\,dt$. Explain why there must be a value x between 0 and 2 such that $F(x) = \pi$.



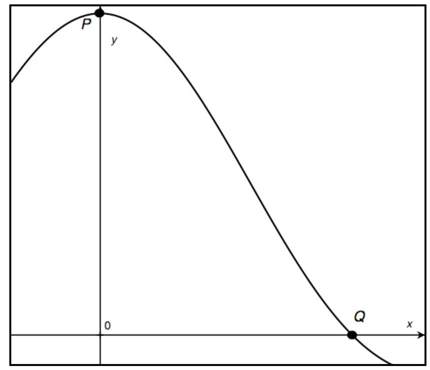
$F(0) = \int_{-2}^{0} f(t)\,dt = 0 \qquad F(2) = \int_{-2}^{2} f(t)\,dt = 4$
F is continuous so by the IVT, since $F(0) < \pi < F(2)$,
there must be a value of x between 0 and 2 such that $F(x) = \pi$.

D. Equation of a Tangent Line

What you are finding: You typically have a function *f* and you are given a point on the function. You want to find the equation of the tangent line to the curve at that point.

How to find it: You use your point-slope equation: $y - y_1 = m(x - x_1)$ where *m* is the slope and (x_1, y_1) is the point. Typically, to find the slope, you take the derivative of the function at the specified point: $f'(x_1)$.

Example 9: Let *f* be the function given by $f(x) = \cos^2 x + \cos x$ as shown in the graph to the right. The curve crosses the *y*-axis at point *P* and the *x*-axis at point *Q*. Find the equation of the tangent line to *f* at *Q*.



To find point Q : $\cos x(\cos x + 1) = 0$

$\cos x = 0, x = \frac{\pi}{2}$, so $Q = \left(\frac{\pi}{2}, 0\right)$

$f'(x) = 2\cos x(-\sin x) - \sin x$

$f'\left(\frac{\pi}{2}\right) = 2\cos\left(\frac{\pi}{2}\right)\left(-\sin\left(\frac{\pi}{2}\right)\right) - \sin\left(\frac{\pi}{2}\right) = -1$

$y - y_1 = m(x - x_1) \Rightarrow y - 0 = -1\left(x - \frac{\pi}{2}\right) \qquad y = \frac{\pi}{2} - x$

Example 10: Suppose that the function *f* has a continuous first derivative for all *x* and that $f(0) = -3$ and $f'(x) = -4$. Let *g* be a function whose derivative is given by $g'(x) = e^{x-1}(2f(x) - 3f'(x))$ for all *x*. Given that $g(0) = 2$, write an equation of the tangent line to the graph of *g* at *x* = 0.

Don't let this problem intimidate you. It still uses the point slope formula.

You are given the point : $(0,2)$

$g'(0) = e^{-1}(2f(0) - 3f'(0)) = \frac{1}{e}[2(-3) - 3(-4)] = \frac{6}{e}$

Tangent line equation : $y - 2 = \frac{6x}{e}$ or $y = \frac{6x}{e} + 2$

Example 11: Consider the differential equation $\frac{dy}{dx} = \sin^{-1}(x^2 - 2y)$. Let $f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 1$. Write an equation for the tangent line to the graph of f at $x = -1$.

This looks like a tough DEQ problem, but it can be used very early in the course.

You have a point : $(-1,1)$ and you have a slope : $\frac{dy}{dx} = \sin^{-1}((-1)^2 - 2(1)) = \sin^{-1}(-1) = \frac{-\pi}{2}$.

So the equation is : $y - 1 = \frac{-\pi}{2}(x + 1)$ No need to simplify this.

Example 12: Consider the curve in the *xy*-plane $x^3 + 4x^2 + y^2 - 2y = 11$. Write an equation for any lines tangent to the curve when $x = -2$.

First, let's find the y value. $-8 + 16 + y^2 - 2y = 11$

$y^2 - 2y = 3 \Rightarrow (y-3)(y+1) = 0 \Rightarrow y = 3, y = -1.$

So the points of tangency are $(-2,3),(-2,-1)$

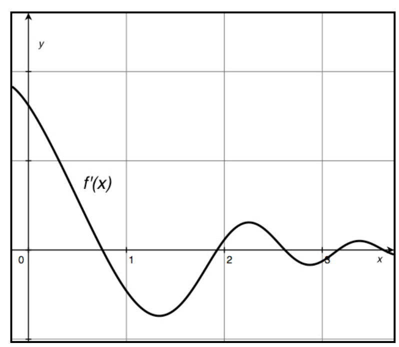
This is a problem in implicit differentiation. The testers may give you the result of the implicit and ask you to verify it. We will determine $\frac{dy}{dx}$.

$$3x^2 + 8x + 2y\frac{dy}{dx} - 2\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 8x = \frac{dy}{dx}(2 - 2y)$$

$$\frac{dy}{dx} = \frac{3x^2 + 8x}{2 - 2y} \qquad \left.\frac{dy}{dx}\right|_{(-2,3)} = 1 \qquad \left.\frac{dy}{dx}\right|_{(-2,-1)} = -1$$

$(-2,3): y - 3 = 1(x+2)$ or $y = x + 5$ $\qquad (-2,-1): y + 1 = -1(x+2)$ or $y = -x - 3$

Example 13: (Calc) Let f be a function defined for $x \geq 0$ with $f(0) = 8$ and f', the derivative of f, is given by $f'(x) = 3e^{-x}\cos(x^2 + 1)$. The graph of f' is shown to the right. Write an equation of the line tangent to the graph of f at $x = 2$.



This would be easy (similar to # 11) if we were asked to write the equation of the tangent line at $x = 0$. But we want the tangent line at $x = 2$.

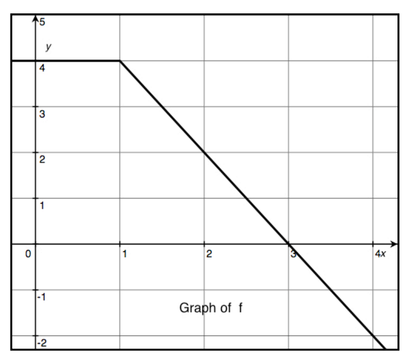
We can find the slope at $x = 2: f'(2) = 3e^{-2}\cos(2^2 + 1) = 0.115$

We know that $f(2) - f(0) = \int_0^2 f'(x)\,dx$ so $f(2) = f(0) + \int_0^2 f'(x)\,dx$

$f(2) = 8 + .092 = 8.092$. Equation of tangent line: $y - 8.092 = .115(x - 2)$

This theme of using the integral of a derivative (rate of change) to get the accumulated change is repeated over and over in AP exams.

Example 14: The graph of the function f to the right consists of two line segments. Let g be the function $g(x) = \int_0^x f(t)\,dt$. Find the equation of the tangent line to g at $x = 4$.



Same method although we don't have the slope or the point.

$$g(4) = \int_0^4 f(t)\,dt = 4 + 4 - 1 = 7.$$

$$g'(x) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x) \qquad \Rightarrow \qquad g'(4) = f(4) = -2$$

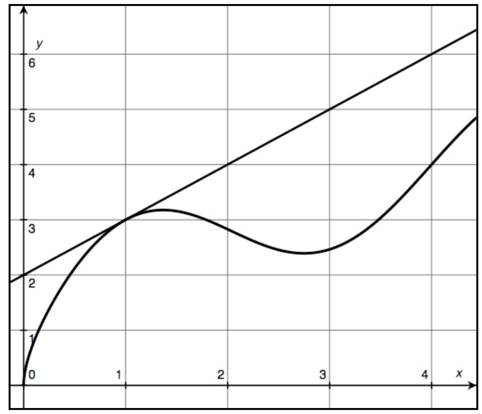
Tangent line: $y - 7 = -2(x - 4)$ or $y = 15 - 2x$

Note that this problem uses accumulation function and 2nd FTC.

E. Local Linear Approximations

What you are finding: You typically have a function f given as a set of points as well as the derivative of the function at those x-values. You want to find the equation of the tangent line to the curve at a value c close to one of the given x-values. You will use that equation to approximate the y-value at c. This uses the concept of local linearity – the closer you get to a point on a curve, the more the curve looks like a line.

How to find it: You use your point-slope equation: $y - y_1 = m(x - x_1)$ where m is the slope and (x_1, y_1) is the point closest to c. Typically, to find the slope, you take the derivative $f'(x_1)$ of the function at the closest x-value given. You then plug c into the equation of the line. Realize that it is an approximation of the corresponding y-value. If 2nd derivative values are given as well, it is possible to determine whether the approximated y-value is above or below the actual y-value by looking at concavity. For instance, if we wanted to approximate $f(1.1)$ for the curve in the graph to the right, we could find $f'(x)$, use it to determine the equation of the tangent line to the curve at x = 1 and then plug 1.1 into that linear equation. If information were given that the curve was concave down, we would know that the estimation over-approximated the actual y-value.



Example 15: Let f be a function that is differentiable for all real numbers. The table below gives the values of f and its derivative f' for selected values in the interval $-0.9 \le x \le 0.9$. The second derivative f'' is always positive in the same closed interval. Write an equation of the line tangent to the graph of f where x = -0.6. Use this line to approximate the value of $f(-0.5)$. Is this approximation greater or less than the actual value of $f(-0.5)$ and give a reason.

x	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
$f(x)$	-34	-87	-99	-100	-84	-51	21
$f'(x)$	-69	-30	-9	0	1	9	90

$y + 87 = -30(x + .6) \Rightarrow y = -30x - 105 \qquad y(-.5) = -30(-.5) - 105 = -90$

Since the slope of the line is negative and the concavity of the curve is positive, the line falls below the curve so the approximation is less than the actual value of $f(-0.5)$

Example 16: Consider the curve $x^2 + \ln\left(\frac{x}{y}\right) = 1$. There is a number k such that the point (1.25, k) is on the curve. Using the tangent line to the curve at x = 1, approximate the value of k.

Solve for y by inspection or: $1 + \ln\left(\frac{1}{y}\right) = 1 \Rightarrow \ln\left(\frac{1}{y}\right) = 0 \Rightarrow e^0 = \frac{1}{y} \Rightarrow y = 1$

$x^2 + \ln x - \ln y = 1$

$2x + \frac{1}{x} - \frac{1}{y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = y\left(2x + \frac{1}{x}\right) \Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = 3$

$y - 1 = 3(x - 1) \Rightarrow y = 3x - 2 \qquad y = 3(1.25) - 2 = 1.75$ so $k \approx 1.75$

F. Continuity and Differentiability

What you are finding: Typical problems ask students to determine whether a function is continuous and/or differentiable at a point. Most functions that are given are continuous in their domain, and functions that are not continuous are not differentiable. So functions given usually tend to be piecewise and the question is whether the function is continuous and also differentiable at the *x*-value where the function changes from one piece to the other.

How to find it: Continuity: I like to think of continuity as being able to draw the function without picking your pencil up from the paper. But to prove continuity at $x = c$, you have to show that $\lim_{x\to c} f(x) = f(c)$.
Usually you will have to show that $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$.
Differentiability: I like to think of differentiability as "smooth." At the value *c*, where the piecewise function changes, the transition from one curve to another must be a smooth one. Sharp corners (like an absolute value curve) or cusp points mean the function is not differentiable there. The test for differentiability at $x = c$ is to show that $\lim_{x\to c^-} f'(x) = \lim_{x\to c^+} f'(x)$. So if you are given a piecewise function, check first for continuity at $x = c$, and if it is continuous, take the derivative of each piece, and check that the derivative is continuous at $x = c$. Lines, polynomials, exponentials and sine and cosines curves are differentiable everywhere.

Example 17: Let $f(x) = \begin{cases} -6x^2 + 14x - 8, x \le 1 \\ \sin(2x-2), x > 1 \end{cases}$. Show that $f(x)$ is differentiable.

Both a parabola and a sine curve are continuous. $f(x)$ is continuous at $x = 1$ because

$\lim_{x\to 1^-} f(x) = -6 + 14 - 8 = 0 \qquad \lim_{x\to 1^+} f(x) = \sin 0 = 0$

$$f'(x) = \begin{cases} -12x + 14, x < 1 \\ 2\cos(2x-2), x \ge 1 \end{cases}$$

Both a line and a cosine curve are continuous. $f(x)$ is differentiable at $x = 1$ because

$\lim_{x\to 1^-} f'(x) = -12 + 14 = 2 \qquad \lim_{x\to 1^+} f'(x) = 2\cos 0 = 2$

Example 18: Find the values of *a* and *b* that make the following function differentiable:

$$y = \begin{cases} ax^3 - 2, x \le 3 \\ b(x-2)^2 + 10, x > 3 \end{cases}$$

$\lim_{x\to 3^-} f(x) = 27a - 2 \qquad \lim_{x\to 3^+} f(x) = b + 10 \Rightarrow \text{For continuity}: 27a = b + 12$

$$f'(x) = \begin{cases} 3ax^2, x \le 3 \\ 2b(x-2), x > 3 \end{cases}$$

$\lim_{x\to 3^-} f'(x) = 27a \qquad \lim_{x\to 3^+} f'(x) = 2b \Rightarrow \text{For differentiability}: 27a = 2b$

$2b = b + 12 \Rightarrow \boxed{b = 12} \qquad 27a = 12 + 12 \Rightarrow \boxed{a = \frac{8}{9}}$

G. Mean Value Theorem (MVT)

What it says: If $f(x)$ is continuous on $[a,b]$ and differentiable on (a, b), there must be some value of c between a and b such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. In words, this says that there must be some value between a and b such that that the tangent line to the function at that value is parallel to the secant line between a and b.

Example 19: The functions f and g are continuous and differentiable for all numbers. The function h is given by $h(x) = f(g(x)) - x$. The table below gives values of the functions. Explain why there must be a value t for $1 < t < 4$ such that $h'(t) = 3$.

x	1	2	3	4
$f(x)$	0	8	-4	6
$g(x)$	3	7	-1	2

$h(1) = f(g(1)) - 1 = f(3) - 1 = -4 - 1 = -5 \qquad h(4) = f(g(4)) - 4 = f(2) - 4 = 8 - 4 = 4$

$\frac{h(4) - h(1)}{4 - 1} = \frac{4 + 5}{3} = 3$

Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value t, $1 < t < 4$, such that $h'(t) = 3$.

Example 20: An amusement park ride travels on a straight track. During the time interval $0 \le t \le 45$ seconds, the vehicle's velocity v, measured in meters/sec, and acceleration a, measured in meters per second per second shows selected values of these functions.

t	0	10	20	25	30	40	45
$v(t)$ (m/sec)	-4.5	-6.2	-7.3	-6.2	-3.0	0	3.0
$a(t)$ (m/sec^2)	2.2	2.5	1.8	0.7	2.2	2.5	1.5

a) For $0 \le t \le 45$, must there be a time t when the vehicle has no acceleration? Justify answer.

Since $v(10) = v(25)$, the MVT guarantees a t in $(10,25)$ such that $v'(t) = a(t) = 0$. This is a special case of the MVT called Rolle's Thereom.

b) For $0 \le t \le 45$, what is the minimum number of times the vehicle is stopped? Explain.

One. Since the speed at 30 seconds is 3 m/sec and and also at 45 seconds, by the MVT there must be some time on (30,45) such that the vehicle is stopped. It is stopped at 40 seconds. It could be stopped at other times on (30,45) but not necessarily.

c) What is the minimum number of times the speed of the vehicle is 1 m/sec? Explain.

Twice. Since the speed at 30 sec. is 3 m/sec and and vehicle is stopped at 40 sec., by the IVT, at some time on $(30,40)$ the vehicle travels at 1 m/sec. Same argument on $(40,45)$.

Example 21: Let g be a twice-differentiable function such that $g(-1) = 9$ and $g(9) = -1$. Let h be the function such that $h(x) = g(g(x))$. Show that $h'(-1) = h'(9)$. Use this result to show why there must be a value c for -1 < c < 9 such that $h''(c) = 0$.

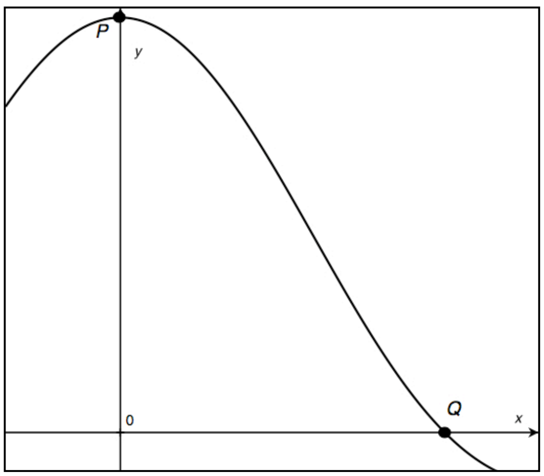
$h'(x) = g'(g(x)) \cdot g'(x)$

$h'(-1) = g'(g(-1)) \cdot g'(-1) = g'(9) \cdot g'(-1)$ $\qquad h'(9) = g'(g(9)) \cdot g'(9) = g'(-1) \cdot g'(9)$ $\qquad$ So $h'(-1) = h'(9)$

Since g is twice differentiable, h is differentiable everywhere.

So by the MVT, there is a value c on $(-1,9)$ such that $h''(c) = \dfrac{h'(9) - h'(-1)}{10} = 0$

Example 22: (Calc) Let f be the function given by $f(x) = \cos^2 x + \cos x$ as shown in the graph below. The curve crosses the y-axis at point P and the x-axis at point Q. Find the value of c guaranteed by the Mean Value Theorem on [0, Q].



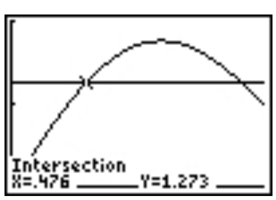
To find point $Q: \cos x(\cos x + 1) = 0$

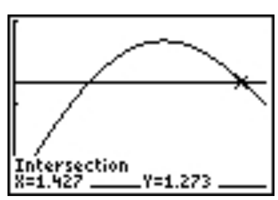
$\cos x = 0, x = \dfrac{\pi}{2}$, so $Q = \left(\dfrac{\pi}{2}, 0\right)$ $\quad P = (0,2)$ $\quad m(\overline{PQ}) = \dfrac{2}{\frac{-\pi}{2}} = \dfrac{-4}{\pi}$

$f'(c) = -2\cos c \sin x - \sin c$

$-2\cos c \sin c - \sin c = \dfrac{-4}{\pi} \Rightarrow 2\sin c \cos c + \sin c = \dfrac{4}{\pi}$

Solving graphically: $c = 0.476$ or $c = 1.427$





H. Horizontal and Vertical Tangent Lines

How to find them: You need to work with $f'(x)$, the derivative of function *f*. Express $f'(x)$ as a fraction.

Horizontal tangent lines: set $f'(x)=0$ and solve for values of *x* in the domain of *f*.

Vertical tangent lines: find values of *x* where $f'(x)$ is undefined (the denominator of $f'(x)=0$).

In both cases, to find the point of tangency, plug in the *x* values you found back into the function *f*. However, if both the numerator and denominator of $f'(x)$ are simultaneously zero, no conclusion can be made about tangent lines. These types of problems go well with implicit differentiation.

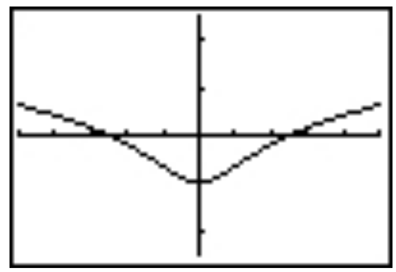
Example 23: (Calc) Let $f''(x)=\dfrac{x}{x^2+1}$ with $f'(0)=-1$. Find the number of horizontal tangent lines the graph of $f(x)$ has on $-5 \le x \le 5$.

$$f'(x)=\int \frac{x}{x^2+1}\,dx=\frac{1}{2}\ln\left(x^2+1\right)+C$$

$$f'(0)=0+C=-1 \Rightarrow C=-1 \qquad f'(x)=\frac{1}{2}\ln\left(x^2+1\right)\ -1$$

By the graph of $f'(x)$, there are 2 values of x where $f'(x)=0$

So the graph of $f(x)$ has 2 horizontal tangents.



Example 24: Let $x^2-2x+y^4+4y=23$. Find the equation of the line(s) that are vertically tangent to $f(x)$.

$$2x-2+\left(4y^3+4\right)\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{1-x}{2\left(y^3+1\right)}$$

Vertical tangent at $y=-1, x\neq 1$

$$x^2-2x+1-4=23 \Rightarrow x^2-2x-26=0$$

$$x=\frac{2\pm\sqrt{4+104}}{2}=\frac{2\pm\sqrt{108}}{2}=1\pm 3\sqrt{3}$$

$$x=1+3\sqrt{3}, x=1-3\sqrt{3}$$

Example 25: Consider the curve $y^3+2x^2y-2x^2+2y+12=0$. Find the equation of the line(s) that are horizontally tangent to $f(x)$.

$$3y^2\frac{dy}{dx}+2x^2\frac{dy}{dx}+4xy-4x+2\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{4x-4xy}{2x^2+3y^2+2}=0$$

$$4x-4xy=0 \Rightarrow 4x(1-y)=0 \Rightarrow x=0, y=1$$

$x=0: y^3+2y+12=0$. By inspection: $y=-2$.

$y=1$ is not a tangent line: $1+2x^2-2x^2+2+12\neq 0$

So $y=-2$ is a horizontal asymptote to the curve.

I. Related Rates

This is a difficult topic to represent. There are many types of questions that could be interpreted as related rates problems. It can be argued that straight-line motion problems fall under the category of related rates. So do the types of questions that interpret a derivative as a rate of change. In this section, we will only focus on questions that use geometric shapes with different quantities changing values.

What you are finding: In related rates problems, any quantities, given or asked to calculate, that are changing are derivatives with respect to time. If you are asked to find how a volume is changing, you are being asked for $\frac{dV}{dt}$. Typically in related rates problems, you need to write the formula for a quantity in terms of a single variable and differentiate that equation with respect to time, remembering that you are actually performing implicit differentiation. If the right side of the formula contains several variables, typically you need to link them somehow. Constants may be plugged in before the differentiation process but variables that are changing may only be plugged in after the differentiation process.

Example 26: Oil is leaking from an oil well at the bottom of a 300-foot deep lake and forms an oil slick that takes the form of a right circular cone. The radius of the slick on the lake's surface is increasing at 10 ft/min at the moment the oil slick is 100 feet wide.

a) Find the rate at which the area of slick is increasing at that time. Indicate units.

$$A = \pi r^2 \Rightarrow \frac{dA}{dT} = 2\pi r \frac{dr}{dT} = 2\pi(50)(10) = 1000\pi \ \frac{\text{ft}^2}{\text{min}}$$

b) Find the rate at which the volume of the slick is changing and express your answer in the proper units. (The volume V of a cone with radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$)

$$V = \frac{1}{3}\pi r^2(300) \qquad \text{as } h \text{ is a constant so } V = 100\pi r^2$$

$$\frac{dV}{dt} = 200\pi r \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 200\pi(50)(10) = 100000\pi \ \frac{\text{ft}^3}{\text{min}}$$

c) A special ship that skims the oil and removes it is dispatched to the scene and starts to remove oil at the moment that the radius of the slick was 100 feet wide. The rate at which oil is removed is $S(t) = 10000\pi\sqrt[3]{t} \ \frac{\text{ft}^3}{\text{min}}$ where t is the time in minutes since the ship starts to remove oil. Using the value you found in part b) as the rate that oil is escaping the well, find the time t when the oil slick reaches its maximum volume.

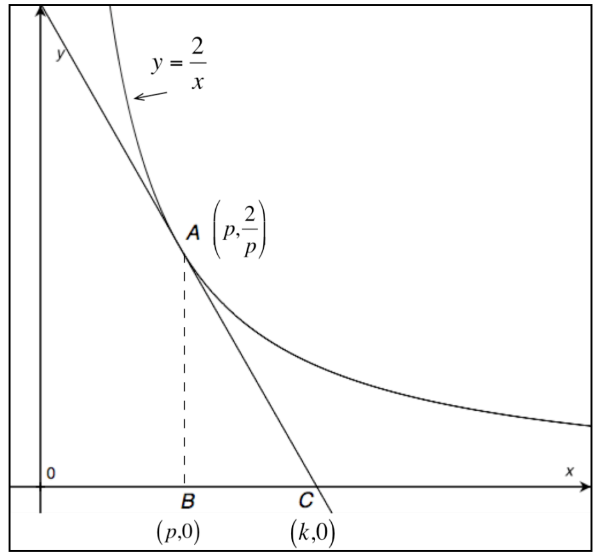
V_1 = volume of slick using skimmer: $\frac{dV_1}{dt} = 100000\pi - S(t) = 0$

$S(t) = 100000\pi \qquad 10000\pi\sqrt[3]{t} = 100000\pi \qquad \sqrt[3]{t} = 10 \Rightarrow t = 1000 \text{ min}$

Since $\frac{dV_1}{dt} > 0$ when $t < 1000$ and $\frac{dV_1}{dt} < 0$ when $t > 1000$

the oil slick reaches maximum volume 1000 (16.67 hrs) minutes after the ship arrives.

Example 27: In the figure to the right, the line is tangent to the graph of $y = \dfrac{2}{x}$ at point A with coordinates $\left(p, \dfrac{2}{p}\right)$. Point B has coordinates $(p, 0)$. The line crosses the x-axis at the point C with coordinates $(k, 0)$.



a) Find k in terms of p.

$$\text{Slope}: \frac{d}{dx}\left(\frac{2}{x}\right) = m = \frac{-2}{x^2} \text{ so } m_p = \frac{-2}{p^2}$$

$$\text{So } \frac{0 - \frac{2}{p}}{k - p} = \frac{-2}{p^2}$$

$$-2p = -2(k - p)$$

$$-2p = -2k + 2p \Rightarrow k = 2p$$

b) Suppose p is increasing at the constant rate of 8 units per second. What is the rate of change of k with respect to time?

$$\frac{dk}{dt} = 2\frac{dp}{dt} = 2(8) = 16 \ \frac{\text{units}}{\text{sec}}$$

c) Suppose p is increasing at a constant rate. Show that there is no change to the area of triangle ABC with respect to time.

$$A = \frac{1}{2}(k - p)\left(\frac{2}{p}\right) = \frac{1}{2}(2p - p)\left(\frac{2}{p}\right) = 1$$

$$\frac{dA}{dt} = 0$$

d) Suppose p is increasing at a constant rate of 4 units/sec. Find the rate of change of hypotenuse AC of the triangle ABC when $p = 2$.

$$h = \sqrt{\left(\frac{2}{p}\right)^2 + (k - p)^2} = \sqrt{\frac{4}{p^2} + (2p - p)^2} = \sqrt{\frac{4}{p^2} + p^2}$$

$$\frac{dh}{dt} = \frac{\frac{-8}{p^3}\frac{dp}{dt} + 2p\frac{dp}{dt}}{2\sqrt{\frac{4}{p^2} + p^2}}$$

$$\frac{dh}{dt} = \frac{\frac{-8}{2^3}(4) + 2(2)(4)}{2\sqrt{\frac{4}{2^2} + 2^2}} = \frac{12}{2\sqrt{5}} = \frac{6}{\sqrt{5}} \ \frac{\text{units}}{\text{sec}}$$

Sometimes, a problem that appears to be a related rates question could be a differential equation in disguise.

Example 28: At time t measured in minutes, the volume of a cube is increasing at a rate proportional to the reciprocal of the length of its side x, measured in inches. At $t = 0$, the side of the cube is 2 inches and at $t = 2$, the side of the cube is 4 inches.

a. Find the side of the cube in terms of t.

$$V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt} = \frac{k}{x}$$
$$3x^3 dx = k\,dt \Rightarrow \int 3x^3 dx = \int k\,dt$$
$$\frac{3x^4}{4} = kt + C \qquad t = 0, x = 2 \Rightarrow C = 12$$
$$\frac{3x^4}{4} = kt + 12 \qquad t = 2, x = 4 \Rightarrow k = 90$$
$$\frac{3x^4}{4} = 90t + 12 \Rightarrow 3x^4 = 360t + 48 \Rightarrow x = \sqrt[4]{120t + 16}$$

b. At what time t will the volume of the cube be 125?

$$x^3 = (120t + 16)^{3/4} = 125$$
$$120t + 16 = 125^{4/3} = 625$$
$$120t = 609$$
$$t = \frac{609}{120} = \frac{203}{40} \text{ sec}$$

c. When the volume of the cube is 125 and growing at $150\frac{\text{in}^3}{\text{min}}$, how fast are the sides growing at that time?

$$125 = x^3 \Rightarrow x = 5$$
$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \Rightarrow 150 = 3(25)\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\frac{\text{in}}{\text{min}}$$

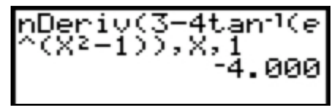
J. Straight-Line Motion – Derivatives

What you are finding: Typically in these problems, you work your way from the position function $x(t)$ to the velocity function $v(t)$ to the acceleration function $a(t)$ by the derivative process. Finding when a particle is stopped involves setting the velocity function $v(t) = 0$. Speed is the absolute value of velocity. Determining whether a particle is speeding up or slowing down involves finding both the velocity and acceleration at a given time. If $v(t)$ and $a(t)$ have the same sign, the particle is speeding up at a specific value of *t*. If their signs are different, then the particle is slowing down.

Example 29: (Calc) A particle's movement along the *x*-axis at time $t \geq 0$ is given by $v(t) = 3 - 4\tan^{-1}\left(e^{t^2-1}\right)$.

a) Find the acceleration of the particle at time $t = 1$.

$a(1) = v'(1) = -4.$
(Since this is a calculator problem, no need to do it analytically.)



b) Is the speed of the particle increasing or decreasing at time $t = 1$. Explain your answer.

$v(1) = -0.142$. Speed is increasing as $v(1) < 0$ and $a(1) < 0$.

c) Find the time $t \geq 0$ at which the particle is farthest to the right. Justify your answer.

$v(t) = 0$ when $3 = 4\tan^{-1}\left(e^{t^2-1}\right)$ or $e^{t^2-1} = \tan(.75)$

$t = \sqrt{\ln(\tan(.75)) + 1} = .964$ is the only critical value for x

$v(t) > 0$ for $0 < t < .964$, $v(t) < 0$ for $t > .964$ so the particle is farthest to the right at $t = 0.964$. Note that the graph of $v(t)$ is not a justification.

Example 30: A particle moves along the *y*-axis with position at time $t \geq 0$ given by $y(t) = \cos\left(\frac{4\pi}{3}t\right)$.

a) Find the acceleration of the particle at time $t = 3$.

$$v(t) = -\frac{4\pi}{3}\sin\left(\frac{4\pi}{3}t\right),\quad a(t) = -\frac{16\pi^2}{9}\cos\left(\frac{4\pi}{3}t\right) \qquad a(3) = -\frac{16\pi^2}{9}\cos(4\pi) = -\frac{16\pi^2}{9}$$

b) Determine if the velocity of the object is decreasing at $t = 3$. Explain your answer.

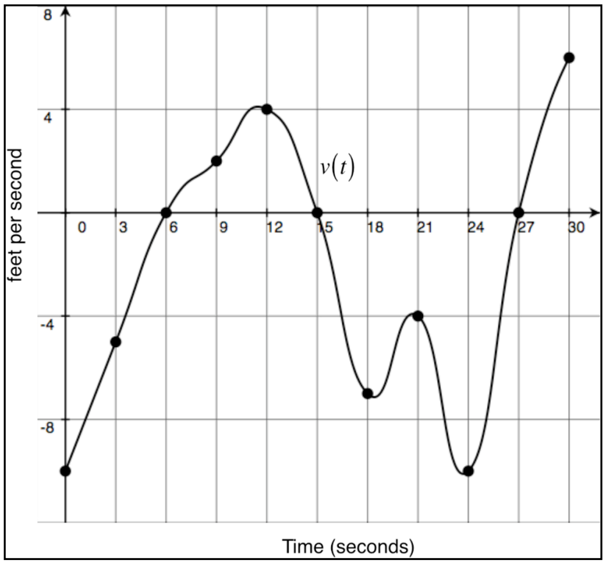
Decreasing velocity is asking whether acceleration < 0 in this interval.
$a(3) = \frac{-16\pi}{9}(\cos 4\pi) = \frac{-16\pi}{9} < 0$. So velocity is decreasing at $t = 3$.

c) Determine if the speed of the object is decreasing at $t = 0.5$. Explain your answer.

$$v\left(\frac{1}{2}\right) = -\frac{4\pi}{3}\sin\left(\frac{4\pi}{3}\cdot\frac{1}{2}\right) = -\frac{4\pi}{3}\sin\left(\frac{2\pi}{3}\right) < 0 \qquad a\left(\frac{1}{2}\right) = -\frac{16\pi^2}{9}\cos\left(\frac{4\pi}{3}\cdot\frac{1}{2}\right) = -\frac{16\pi^2}{9}\cos\left(\frac{2\pi}{3}\right) > 0$$

Since the velocity is negative and acceleration is positive, the speed is decreasing.
Note that this problems uses trig functions of special angles which is fair game.

Example 31: A searchlight is shining along the straight wall of a prison. A graph of the velocity of the light, $v(t)$ at 3-second intervals of time t is shown in the table as well as a table of values.



t (seconds)	$v(t)$ ft per second
0	-10
3	-5
6	0
9	2
12	4
15	0
18	-7
21	-4
24	-10
27	0
30	6

a) At what interval of times is the acceleration of the searchlight positive? Why?

$a(t) > 0$ on $(0,12),(18,21),(24,30)$ because $v(t)$ is increasing on those intervals.

b) Find the average acceleration of the searchlight over the interval $9 \le t \le 27$. Express units.

$$\text{Avg. Acc.} = \frac{v(27) - v(9)}{27 - 9} = \frac{0 - 2}{18} = -\frac{1}{9} \text{ ft/sec}^2$$

c) Give an approximation for the acceleration of the searchlight at $t = 15$. Show the computation you used to arrive at your answer.

$$a(15) \approx \frac{v(18) - v(15)}{18 - 15} = \frac{-7 - 0}{3} = \frac{-7}{3} \text{ ft/sec}^2 \text{ or}$$
$$a(15) \approx \frac{v(15) - v(12)}{15 - 12} = \frac{0 - 4}{3} = \frac{-4}{3} \text{ ft/sec}^2 \text{ or}$$
$$a(15) \approx \frac{v(18) - v(12)}{18 - 12} = \frac{-7 - 4}{6} = \frac{-11}{6} \text{ ft/sec}^2$$

d) For what values of t is the searchlight speeding up on the wall? Justify your answers.

$(6,12),(15,18),(21,24)$ and $(27,30)$
because $v(t)$ and $a(t)$ have the same signs in those intervals.