R. Average Value of a Function

What you are finding: You are given a continuous function $f(x)$, either an algebraic formula or a graph, as well as an interval $[a,b]$. You wish to find the average value of the function on that interval.

How to find it: $f_{avg} = \dfrac{\int_a^b f(x)\,dx}{b-a}$. The units will be whatever the function *f* is measured in.

Again, be careful. The average rate of change of a function *F* on $[a,b]$ is not the same as the average value of the function *F* on $[a,b]$.

Average value of the function: $\dfrac{\int_a^b f(x)\,dx}{b-a}$

Average rate of change of the function (average value of the rate of change): $\dfrac{\int_a^b F'(x)\,dx}{b-a} = \dfrac{F(b)-F(a)}{b-a}$

Example 55: If $f(x) = x^2 - 2x + 3$, find

a) the average rate of change of *f* on [-3, 3]:

$$\frac{\int_{-3}^{3}(2x-2)\,dx}{3+3} = \frac{\left(x^2-2x\right)\Big|_{-3}^{3}}{6} = \frac{3-15}{6} = -2 \quad \text{or (better)} \quad \frac{f(3)-f(-3)}{3+3} = \frac{6-18}{6} = -2$$

b) the average value of *f* on [-3, 3]:

$$f_{avg} = \frac{\int_{-3}^{3}\left(x^2-2x+3\right)dx}{3+3} = \frac{\left(\frac{x^3}{3}-x^2+3x\right)\Big|_{-3}^{3}}{6} = \frac{9+27}{6} = 6$$

Example 56: (Calc) If $f(t) = \ln\left(t^2+2t-2\right)$, find

a) the average rate of change of *f* on [1, *e*]:

$$\frac{\int_{1}^{e}\frac{d}{dt}\left[\ln\left(t^2+2t-2\right)\right]dt}{e-1} \quad \text{or (better)} \quad \frac{f(e)-f(1)}{e-1} = \frac{\ln\left(e^2+2e-2\right)-\ln(1+2-2)}{e-1} = 1.386$$

b) the average value of *f* on [1, *e*]:

$$f_{avg} = \frac{\int_{1}^{e}\ln\left(t^2+2t-2\right)dt}{e-1} = 1.517$$

Example 57: A particle moves along the x-axis such that its velocity is given by $v(t) = t\cos(t^2)$. The particle is at position $x = 3$ at time $t = 0$. Assume v is measured in feet and t is measured in seconds.

a) Find the average speed of the particle over the interval $0 \le t \le \sqrt{\pi/2}$. Specify units.

For $0 \le t \le \sqrt{\pi/2}$, $t\cos(t^2) > 0$ so speed = velocity

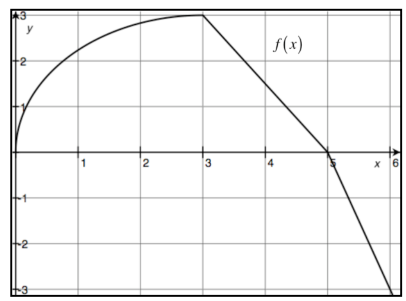
$$\text{Avg speed} = \frac{\int_0^{\sqrt{\pi/2}} t\cos(t^2)\,dt}{\sqrt{\pi/2} - 0} = \left.\frac{\sin(t^2)}{2\sqrt{\pi/2}}\right|_0^{\sqrt{\pi/2}} = \frac{1}{\sqrt{2\pi}}\ \frac{\text{ft}}{\text{sec}}$$

b) Find the average acceleration of the particle over the interval $0 \le t \le \sqrt{\pi}$.

$$\text{Avg accel} = \frac{\int_0^{\sqrt{\pi}} a(t)\,dt}{\sqrt{\pi} - 0} = \frac{\int_0^{\sqrt{\pi}} v'(t)\,dt}{\sqrt{\pi}} = \left.\frac{v(\sqrt{\pi}) - v(0)}{\sqrt{\pi}}\right| = \frac{-\sqrt{\pi}}{\sqrt{\pi}} = -1\ \frac{\text{ft}}{\text{sec}^2}$$

Note that the initial position of the particle is irrelevant.

Example 58: The graph of $f(x)$ is comprised of a quarter circle and 2 lines as shown in the figure on the right.



a) Find the average value of $f(x)$ on [0, 6].

$$f_{avg} = \frac{\int_0^6 f(x)\,dx}{6} = \frac{\frac{9\pi}{4} + 3 - \frac{3}{2}}{6} = \frac{\frac{9\pi}{4} + \frac{3}{2}}{6} = \frac{3\pi + 2}{8}$$

b) Suppose the straight line between $x = 5$ and $x = 6$ in the graph continues. Write, but do not solve, an equation to find the value of k such that the average value of $f(x)$ on [0, k] is zero.

$$\frac{\int_0^5 f(x)\,dx}{5} + \frac{\int_5^k f(x)\,dx}{k-5} = 0 \quad \text{or} \quad \frac{\int_0^5 f(x)\,dx}{5} = -\frac{\int_5^k f(x)\,dx}{k-5}$$

Example 59: The "Laughing Gas" company rents helium tanks in the shape of a cylinder. Their tanks are all 2 feet long but have variable radii, from 2 to 6 inches. The dollar cost of renting a tank is

$$C = 20 + \frac{\text{Volume of cylinder in inches}^3}{100\ \text{inches}^3}$$

a) Find the average cost of all the tanks that the company rents.

$$C_{avg} = \frac{\int_2^6 \left(20 + .24\pi R^2\right) dR}{6-2} = \frac{132.28}{4} = \$33.07$$

b) Find the average rate of change of the cost as one changes from a 2 inch radius tank to a 6 inch radius tank. Use appropriate units.

$$\frac{C(6) - C(2)}{6-2} = \frac{47.14 - 23.02}{4} = \frac{\$6.03}{\text{inch}}$$

Average value problems can cause some confusion. Let us look at a slightly different type of problem:

Example 60: Jamie takes on a job to provide four hours of babysitting. She agrees on a price of \$40 for the four hours.

a) What is her average pay per hour?

$\boxed{\text{Certainly a question that a grade - school student can answer. \$10 an hour.}}$

Suppose that Jamie is paid at the **end** of the 4 hour session. Let's think of her payment in terms of how much additional money is in her wallet at the end of every hour.

Hour (*t*)	0	1	2	3	4
Money (*M*)	0	0	0	0	40

b) Calculate $\int_0^4 M(t)\,dt$ and $\frac{1}{4}\int_0^4 M(t)\,dt$ using right Riemann sums and interpret their meaning using proper units.

$\int_0^4 M(t)\,dt = 40$. This would be measured in money - hours. This really has little meaning.

$\frac{1}{4}\int_0^4 M(t)\,dt = 10$. The average amount of money in her wallet over the 4 hours is \$10.

c) Calculate $\int_0^4 M(t)\,dt$ and $\frac{1}{4}\int_0^4 M(t)\,dt$ using left Riemann sums and interpret their meaning using proper units assuming that Jamie gets paid at the **beginning** of the 4-hour session.

Hour (*t*)	0	1	2	3	4
Money (*M*)	40	40	40	40	40

$\int_0^4 M(t)\,dt = 160$. This would be measured in money - hours.

$\frac{1}{4}\int_0^4 M(t)\,dt = 40$. The average amount of money in her wallet over the 4 hours is \$40.

d) Calculate $\int_0^4 M(t)\,dt$ and $\frac{1}{4}\int_0^4 M(t)\,dt$ using right Riemann sums and interpret the meaning using proper units assuming that Jamie gets paid **halfway through** the 4-hour session.

Hour (*t*)	0	1	2	3	4
Money (*M*)	0	0	40	40	40

$\int_0^4 M(t)\,dt = 80$. This would be measured in money - hours.

$\frac{1}{4}\int_0^4 M(t)\,dt = 20$. The average amount of money in her wallet over the 4 hours is \$20.

Example 61: (Calc) A tennis tournament is free for people to watch. People line up at the gate waiting for it to open in order to get the best seats. . The number of people in line is modeled by a differentiable function L for $0 \le t < 60$ where t is measured in minutes. Values of $L(t)$ at various times t are shown in the table below.

t (minutes)	0	10	20	30	45	60
$L(t)$ (people in line)	60	80	120	200	350	600

a) Find the average rate of people lining up from $t = 0$ to $t = 60$. Specify units.

$$\frac{L(60)-L(0)}{60-0} = \frac{600-60}{60} = 9 \text{ people per minute.}$$

b) Approximate the rate in which people are lining up at $t = 60$ minutes.

$$L'(60) \approx \frac{L(60)-L(45)}{60-45} = \frac{600-350}{15} = 16.667 \text{ people per minute.}$$

c) Using the answer from part b), estimate the number of people in line at $t = 55$ minutes. Show your reasoning.

$$L - L_{60} = L'(60)(t-60)$$
$$L - 600 = 16.667(t-60) \Rightarrow L = 16.667t - 400$$
$$L(55) \approx 16.667(55) - 400 \approx 517 \text{ people}$$

d) Using a trapezoidal sum with the five subintervals given by the table, approximate the value of $\int_0^{60} L(t)\,dt$. Explain the meaning of $\int_0^{60} L(t)\,dt$ in terms of people in line.

$$\int_0^{60} L(t)\,dt \approx \frac{10}{2}(60+80) + \frac{10}{2}(80+120) + \frac{10}{2}(120+200) + \frac{15}{2}(200+350) + \frac{15}{2}(350+600)$$
$$\int_0^{60} L(t)\,dt \approx 700 + 1000 + 1600 + 4125 + 7125 = 14550.$$

This is the total number of minutes people have waited in line over the 60-minute time period.

e) Using the answer from part d), approximate the value of $\frac{1}{60}\int_0^{60} L(t)\,dt$. Explain the meaning of $\frac{1}{60}\int_0^{60} L(t)\,dt$ in terms of people in line.

$$\frac{1}{60}\int_0^{60} L(t)\,dt \approx \frac{14450}{60} = 242.5$$

This is the average number of people in line over the 60-minute time period.

S. Straight-Line Motion - Integrals

What you are finding: In section J, we looked at straight-line motion by the derivative process. You were typically given a position function $x(t)$ and took its derivative to find the velocity function $v(t)$ and took the velocity's derivative to get the acceleration function $a(t)$. Questions like finding maximum velocity or minimum acceleration could then be answered.

Using integrals, we can work backwards. Typical questions involve knowing the velocity function, the position of the particle at the start of the problem, and integrating to find the position function and then finding when the particle is farthest to the right or left.

How to find it: $x(t) = \int v(t)\,dt + C$ and $v(t) = \int a(t)\,dt + C$. Using these equations is essentially solving DEQ's with an initial condition, usually the position at time $t = 0$ or velocity at $t = 0$. Two other concepts that come into play are displacement and distance over some time interval $[t_1,t_2]$.

<u>Displacement:</u> the difference in position over $[t_1,t_2]$: $x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)\,dt$. Displacement can be positive, negative or zero.

<u>Distance</u>: how far the particle traveled over $[t_1,t_2]$: $\int_{t_1}^{t_2} |v(t)|\,dt$. Distance is always positive.

Example 62: A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $3t^2 - 5t - 2$. The position of the particle at time $t = 2$ is $x = 5$.

a) Write a polynomial expression for the position of the particle at any time $t \geq 0$.

$$x(t) = \int (3t^2 - 5t - 2)dt = t^3 - \frac{5t^2}{2} - 2t + C$$

$$x(2) = 8 - 10 - 4 + C = 5 \Rightarrow C = 11 \qquad \Rightarrow \quad x(t) = t^3 - \frac{5t^2}{2} - 2t + 11$$

b) Find the displacement of the particle from time $t = 0$ until time $t = 4$.

Since we have the position function: Disp. $= x(4) - x(0) = 64 - 40 - 8 + 11 - 11 = 16$

If we did not have the position function: Disp. $= \int_0^4 (3t^2 - 5t - 2)dt = \left(t^3 - \frac{5t^2}{2} - 2t\right)\Big|_0^4 = 16$

Note that this question is not dependent on knowng the original position of the particle.

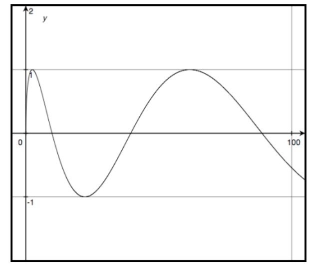
c) Find the distance that the particle moves from time $t = 0$ until time $t = 4$.

Distance $= \int_0^4 \left|(3t^2 - 5t - 2)\right|dt$ Use calculators if allowed. If not, find the values of t such that $v(t) < 0$. To do that, you need to set $v(t) = 0 \Rightarrow 3t^2 - 5t - 2 = 0 \Rightarrow (3t+1)(t-2) = 0$.

So $v(t) < 0$ on $[0,2]$ (particle moving left) and $v(t) > 0$ on $[2,4]$ (particle moving right)

$$\int_0^4 \left|(3t^2 - 5t - 2)\right|dt = -\int_0^2 (3t^2 - 5t - 2)\,dt + \int_2^4 (3t^2 - 5t - 2)\,dt = -\left(t^3 - \frac{5t^2}{2} - 2t\right)\Big|_0^2 + \left(t^3 - \frac{5t^2}{2} - 2t\right)\Big|_2^4 = 28$$

Example 63: (Extension of Ex. 46) (Calc) A particle moves along the x-axis such that its velocity v at time $t \geq 0$ is given by $v(t) = \sin\sqrt{t}$ for $0 \leq t \leq 100$. The graph of $v(t)$ is shown to the right. The position of the particle at time t is $x(t)$ and its position at time $t = 0$ is $x(0) = 10$.



a) Find the distance traveled by the particle from $t = 0$ to $t = 100$.

$$v(t) = 0 \Rightarrow \sin\sqrt{t} = 0 \qquad \sqrt{t} = 0, \pi, 2\pi, 3\pi \text{ so } t = 0, \pi^2, 4\pi^2, 9\pi^2$$

$$v(t) > 0 \text{ if } 0 < t < \pi^2 \text{ and } 4\pi^2 < t < 9\pi^2$$

$$\text{Dist} = \int_0^{\pi^2} \sin\sqrt{t}\, dt - \int_{\pi^2}^{4\pi^2} \sin\sqrt{t}\, dt + \int_{4\pi^2}^{9\pi^2} \sin\sqrt{t}\, dt - \int_{9\pi^2}^{100} \sin\sqrt{t}\, dt = 59.705$$

$$\text{Easier}: \int_0^{100} \left|\sin\sqrt{t}\right| dt = 59.705$$

b) Find the position of the particle at time $t = 100$.

$$x(100) = x(0) + \int_0^{100} \sin\sqrt{t}\, dt = 10 + 15.693 = 26.693$$

Example 64: (**Extension of Ex. 31)** A searchlight is shining along the straight wall of a prison. A graph of the velocity of the light, $v(t)$ at 3-second intervals of time t is shown in the table as well as a table of values.

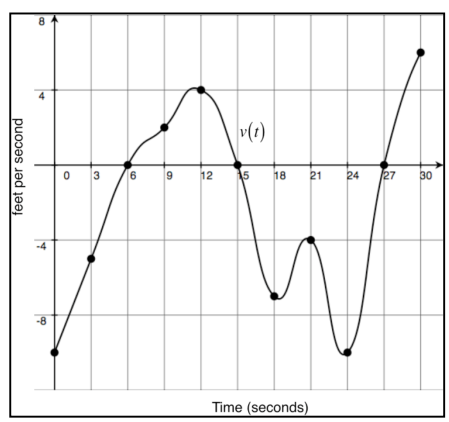
t (seconds)	$v(t)$ ft per second
0	-10
3	-5
6	0
9	2
12	4
15	0
18	-7
21	-4
24	-10
27	0
30	6

a) Using a midpoint Riemann sum with 5 rectangles, approximate the position of the searchlight at $t = 30$ seconds from where it was at $t = 0$.

$$\text{Disp} \approx 6(-5 + 2 + 0 - 4 + 0) = -42$$

Light is approximately 42 feet to the left of where it started.

b) Using the trapezoidal method with 10 trapezoids, approximate how far the searchlight traveled in 30 seconds.



$$\text{Dist} \approx \frac{3}{2}(10 + 10 + 0 + 4 + 8 + 0 + 14 + 8 + 20 + 0 + 6) = 120$$

Light traveled approximately 120 feet.

T. Area/Volume Problems

What you are finding: Typically, these are problems with which students feel more comfortable because they are told exactly what to do or to find. Area and volume are lumped together because, almost always, they are both tested within the confines of a single A.P. free response question. Usually, but not always, they are on the calculator section of the free-response section.

Area problems usually involve finding the area of a region under a curve or the area between two curves between two values of *x*. Volume problems usually involve finding the volume of a solid when rotating a curve about a line.

How to find it: Area: Given two curves $f(x)$ and $g(x)$ with $f(x) \geq g(x)$ on an interval [*a*, *b*], the area between *f* and *g* on [*a*, *b*] is given by $A = \int_{x=a}^{x=b} \left[f(x) - g(x) \right] dx$. While integration is usually done with respect to the *x*-axis, these problems sometimes show up in terms of *y*; $A = \int_{y=c}^{y=d} \left[m(y) - n(y) \right] dy$.

Volume: Disks and Washers: The method I recommend is to establish the outside Radius *R*, the distance from the line of rotation to the outside curve, and, if it exists, the inside radius *r*, the distance from the line of rotation to the inside curve. The formula when rotating these curves about a line on an interval is given by:

$$V = \pi \int_{x=a}^{x=b} \left(\left[R(x) \right]^2 - \left[r(x) \right]^2 \right) dx \quad \text{or} \quad V = \pi \int_{y=c}^{y=d} \left(\left[R(y) \right]^2 - \left[r(y) \right]^2 \right) dy$$

A favorite type of problem is creating a solid with the region *R* being the base of the solid. Cross sections perpendicular to an axis are typically squares, equilateral triangles, right triangles, or semi-circles. Rather than give formulas for this, it is suggested that you draw the figure, establish its area in terms of *x* or *y*, and integrate that expression on the given interval.

Example 65: (Calc) Let *R* be the region in the first quadrant bounded by the graphs of $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$.

a) Find the area of *R*.

$$A = \int_0^8 \left(\sqrt[3]{x} - \frac{x}{4} \right) dx = 4 \quad \text{or} \quad A = \int_0^2 \left(4y - y^3 \right) dx = 4$$

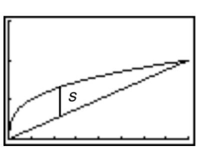
b) Find the volume of the solid when *R* is rotated about the *x*-axis.

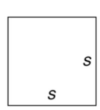
$$V = \pi \int_0^8 \left(x^{2/3} - \frac{x^2}{16} \right) dx = 8.533\pi$$

c) Find the volume of the solid when *R* is rotated about the *y*-axis.

$$V = \pi \int_0^2 \left(16y^2 - y^6 \right) dx = 24.381\pi$$

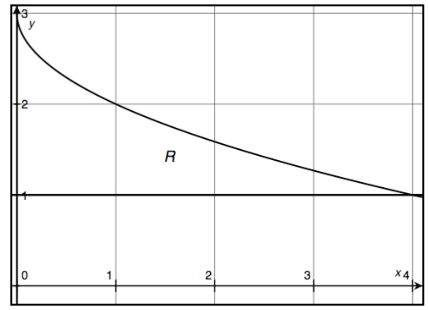
d) The region *R* is the base of a solid. For this solid, the cross sections perpendicular to the *x*-axis are squares. Find the volume of this solid.





$$V = \int_0^8 \left(\sqrt[3]{x} - \frac{x}{4} \right)^2 dx = 2.438. \quad \text{No } \pi \text{ in this!}$$

Example 66: Let R be the region in the first quadrant bounded by the graphs of $y = 3 - \sqrt{x}$, the horizontal line $y = 1$, and the y-axis as shown in the figure to the right.



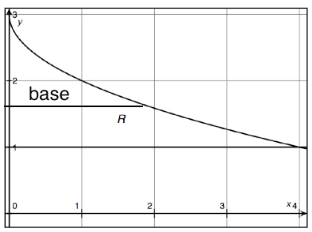
a) Find the area of R.

$$A = \int_0^4 \left(3 - \sqrt{x} - 1\right) dx = \left(2x - \frac{2}{3}x^{3/2}\right)\Bigg|_0^4 = 8 - \frac{2}{3}(8) = \frac{8}{3}$$

b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -1$.

$$V = \pi\int_0^4 \left[\left(3 - \sqrt{x} - (-1)\right)^2 - \left(1 - (-1)\right)^2\right] dx = \pi\int_0^4 \left[\left(4 - \sqrt{x}\right)^2 - 4\right] dx$$

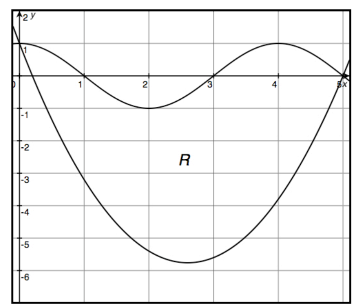
c) Region R is the base of a solid. For each y, where $1 \le y \le 3$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is half the length of its base. Write, but do not evaluate, an integral expression that gives the volume of the solid.



base $= x = (3 - y)^2$ height $= \dfrac{(3-y)^2}{2}$

$$V = \int_1^3 \frac{(3-y)^4}{2}\, dy$$

Example 67: (Calc) Let R be the region in the first quadrant bounded by the graphs of $y = \cos\left(\frac{\pi x}{2}\right)$ and $y = x^2 - \frac{26}{5}x + 1$, shown in the figure.



a) Find the area of R.

$$A = \int_0^5 \left[\cos\left(\frac{\pi x}{2}\right) - \left(x^2 - \frac{26}{5}x + 1\right)\right] dx = 18.970$$

b) The vertical line $x = k$ splits the region R into two equal parts. Write, but do not solve, an equation involving integrals that solves for k.

$$\int_0^k \left[\cos\left(\frac{\pi x}{2}\right) - \left(x^2 - \frac{26}{5}x + 1\right)\right] dx = \int_k^5 \left[\cos\left(\frac{\pi x}{2}\right) - \left(x^2 - \frac{26}{5}x + 1\right)\right] dx$$

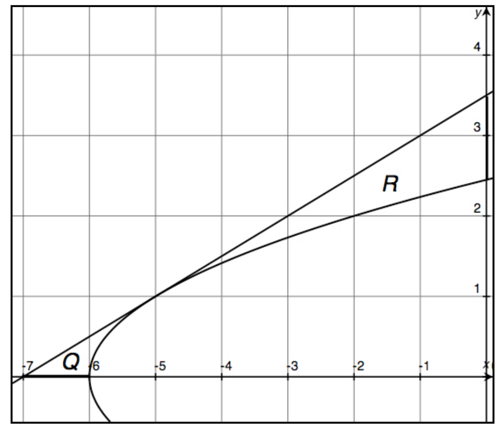
c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

$$V = \int_0^5 \left[\cos\left(\frac{\pi x}{2}\right) - \left(x^2 - \frac{26}{5}x + 1\right)\right]^2 dx = 84.317$$

d) The region R models the surface of a piece of glass. At all points in R at a distance x from the y-axis, the thickness of the glass is given by $h(x) = 5 - x$. Find the volume of the glass.

$$V = \int_0^5 \left[\cos\left(\frac{\pi x}{2}\right) - \left(x^2 - \frac{26}{5}x + 1\right)\right](5 - x)\, dx = 44.155$$

Example 68: (Calc) Line l is tangent to $y^2 = x + 6$ at the point (-5, 1). Let R be the region bounded by $y^2 = x + 6$, line l, and the y-axis. Let Q be the region bounded by $y^2 = x + 6$, line l, and the x-axis. The figure to the right shows these regions.



a) Show that the equation of line l is $y = \dfrac{x+7}{2}$.

$$2y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y} \qquad \left.\frac{dy}{dx}\right|_{(-5,1)} = \frac{1}{2}$$
$$y - 1 = \frac{1}{2}(x+5) \Rightarrow y = \frac{x+7}{2}$$

b) Find the area of region R.

$$A = \int_{-5}^{0}\left(\frac{x+7}{2} - \sqrt{x+6}\right)dx = 2.119$$

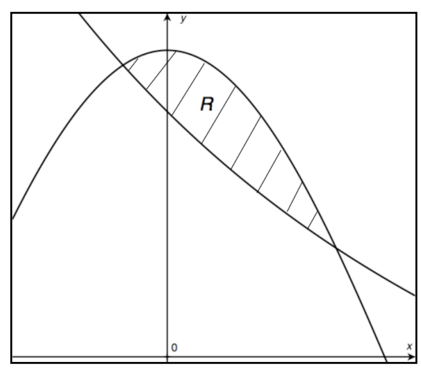
c) Find the volume of the solid generated by revolving region R about the line $y = 1$.

$$V = \pi\int_{-5}^{0}\left[\left(\frac{x+7}{2} - 1\right)^2 - \left(\sqrt{x+6} - 1\right)^2\right]dx = 6.179\pi$$

d) Find the area of region Q.

$$A = \int_{-7}^{-6}\left(\frac{x+7}{2}\right)dx + \int_{-6}^{-5}\left(\frac{x+7}{2} - \sqrt{x+6}\right)dx = .333 \quad \text{or} \quad A = \int_{0}^{1}\left[y^2 - 6 - (2y - 7)\right]dy = .333$$

Example 69: (Calc) Let R be the region bounded by the graphs of $f(x) = 6\cos(2x) - 1$ and $g(x) = 4e^{-x} - x$.



a) Find the area of R.

Curves intersect at S $(x = -0.143)$ and T $(x = 0.545)$

$$A = \int_{S}^{T}\left[6\cos(2x) - 1 - \left(4e^{-x} - x\right)\right]dx = 0.661$$

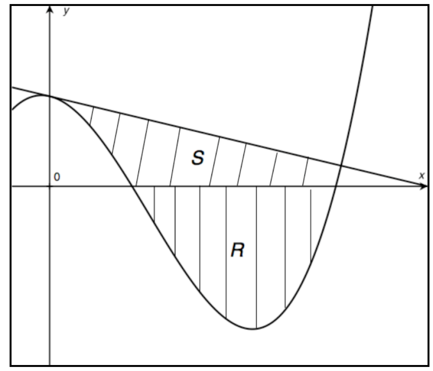
b) Find the volume of the solid generated when R is revolved about the x-axis.

$$V = \pi\int_{S}^{T}\left[\left(6\cos(2x) - 1\right)^2 - \left(4e^{-x} - x\right)^2\right]dx = 4.915\pi = 15.441$$

c) The region R is the base of a solid with cross sections perpendicular to the x-axis as semi-circles with diameters extending from $y = f(x)$ to $y = g(x)$. Find the volume of the solid.

$$V = \frac{1}{2}\pi\int_{S}^{T}\left[\frac{f(x) - g(x)}{2}\right]^2 dx = \frac{\pi}{2}\int_{S}^{T}\left[\frac{6\cos(2x) - 1 - 4e^{-x} + x}{2}\right]^2 dx = 0.301$$

Example 70: **(Calc)** Let $f(x) = x^3 - 2x^2 - \frac{x}{3} + \cos x$ and line l be the line tangent to the graph of f at $x = 0$. Let R be the 4th quadrant region bounded by f and the x-axis and let S be the first quadrant area bounded by f, l, and the x-axis.



a) Find the area of region R.

The positive roots of $f(x)$ are 0.654 and 2.272.
$$A = \int_{.654}^{2.272} \left[0 - \left(x^3 - 2x^2 - \frac{x}{3} + \cos x\right)\right] dx = 1.650$$

b) Find the volume of the solid created by revolving region R about the x-axis.

$$V = \pi \int_{.654}^{2.272} \left(x^3 - 2x^2 - \frac{x}{3} + \cos x\right)^2 dx = 2.065\pi$$

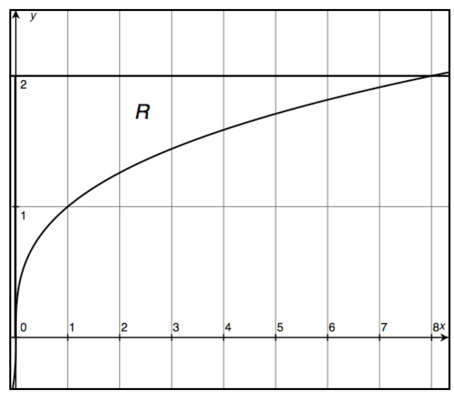
c) Find the area of region S.

$f'(x) = 3x^2 - 4x - \frac{1}{3} - \sin x \Rightarrow f'(0) = \frac{-1}{3}$ Line $l : y = 1 - \frac{x}{3}$, intersection of $f(x)$ and $l : x = 2.313$

$$\text{Area } S + R = \int_{0}^{2.313} \left[1 - \frac{x}{3} - \left(x^3 - 2x^2 - \frac{x}{3} + \cos x\right)\right] dx = 2.670$$

$\text{Area } S = 2.670 - \text{Area}(R) = 2.670 - 1.650 = 1.020$

Example 71: Let R be the region bounded by the graph of $y = \sqrt[3]{x}$, the y-axis, and $y = 2$ as shown in the figure to the right.



a) Find the area of Region R.

$$A = \int_0^8 \left(2 - x^{1/3}\right) dx = \left(2x - \frac{3x^{4/3}}{4}\right)\Big|_0^8 = 16 - 12 = 4$$

$$\text{or } A = \int_0^2 y^3\, dy = \left(\frac{y^4}{4}\right)\Big|_0^2 = 4$$

b) Find the volume of the solid if region R is rotated about the y-axis.

$$V = \pi \int_0^2 \left(y^3\right)^2 dy = \left(\frac{y^7}{7}\right)\Big|_0^2 = \frac{128}{7}\pi$$

c) Let the solid in part b) model a vase with x,y measured in inches. The vase fills with water at the rate of $\frac{6 \text{ in}^3}{\text{min}}$. When the water is 1-inch high, how fast does the water rise in the vase?

Let k represent the height of the water: $V = \pi \int_0^k \left(y^3\right)^2 dy = \left(\frac{y^7}{7}\right)\Big|_0^k = \frac{\pi k^7}{7}$

$\frac{dV}{dt} = \pi k^6 \frac{dk}{dt} \Rightarrow \text{When } k = 1: 6 = \pi \frac{dk}{dt} \Rightarrow \frac{dk}{dt} = \frac{6}{\pi} \frac{\text{in}}{\text{min}}$

U. Derivative of an Inverse of a Function

What you are finding: There is probably no topic that confuses students (and teachers) more than inverses. The inverse of a function *f* is another function f^{-1} that "undoes" what *f* does. So $f^{-1}\left(f(x)\right) = x$. For instance, the inverse of adding 5 is subtracting 5. Start with any number *x*, add 5, then subtract 5, and you are back to *x*. Do not confuse the inverse f^{-1} with the reciprocal. $x^{-1} = \frac{1}{x}$ but $\left(f(x)\right)^{-1} \neq \frac{1}{f(x)}$.

To find the inverse of a function, you replace *x* with *y* and *y* with *x*. The inverse to the function $y = 4x - 1$ is $x = 4y - 1$ or $y = \frac{x+1}{4}$.

In this section, you are concerned with finding the derivative of the inverse to a function: $\frac{d}{dx}\left[f(x)\right]^{-1}$

How to find it: The formula used is: $\frac{dy}{dx} = \frac{1}{f'(y)}$. But I suggest, rather than memorizing this formula, is to switch *x* and *y* to find the inverse, and then take the derivative, using implicit differentiation:

$$x = f(y) \Rightarrow 1 = f'(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)}.$$

Example 72: Find the derivative of the inverse to $y = x^3$ at $x = 1$.

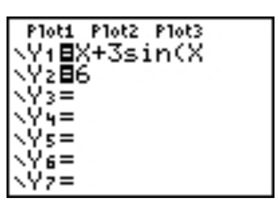
$$\text{Inverse}: x = y^3 = 1 \Rightarrow y = 1$$
$$1 = 3y^2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{3y^2} \Rightarrow \left.\frac{dy}{dx}\right|_{y=1} = \frac{1}{3}$$

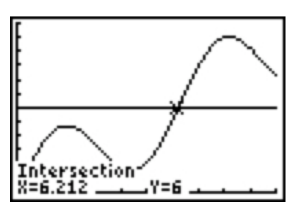
Example 73: Find the derivative of the inverse to $f(x) = 3x + \sin\pi x$ at $x = 6$.

$$\text{Inverse}: x = 3y + \sin\pi y = 6 \Rightarrow y = 2 \quad (\text{usually, these are solved by inspection})$$
$$1 = \left(3 + \pi\cos\pi y\right)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{3 + \pi\cos\pi y} \Rightarrow \left.\frac{dy}{dx}\right|_{y=2} = \frac{1}{3 + \pi\cos 2\pi} = \frac{1}{3+\pi}$$

Example 74: (Calc) Find the derivative of the inverse to $f(x) = x + 3\sin x$ at $x = 6$.

$$\text{Inverse}: x = y + 3\sin y = 6$$
On the calculator, find the value of x where $x + 3\sin x = 6$. That is the y - value.
$$1 = \left(1 + 3\cos y\right)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{1 + 3\cos y} \Rightarrow \left.\frac{dy}{dx}\right|_{y=6.212} = .250$$





Example 75: If $f(x)=x^2$ (1[st] quadrant), write the equation of the tangent line to $f^{-1}(x)$ at $x = 16$.

$$\text{Inverse}: x = y^2 = 16 \Rightarrow y = 4$$
$$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{y=4} = \frac{1}{8}$$
$$\text{So equation of tangent line}: y-4 = \frac{1}{8}(x-16) \text{ or } y = \frac{x}{8}+2$$

Example 76: If $f(x)=x^3+x^2+x+1$, write the equation of the tangent line to $f^{-1}(x)$ at $x = 4$.

$$\text{Inverse}: x = y^3+y^2+y+1 = 4 \Rightarrow y = 1 \quad (\text{by inspection})$$
$$1 = (3y^2+2y+1)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{3y^2+2y+1} \Rightarrow \left.\frac{dy}{dx}\right|_{y=1} = \frac{1}{6}$$
$$\text{So equation of tangent line}: y-1 = \frac{1}{6}(x-4) \text{ or } y = \frac{x}{6}+\frac{1}{3}$$

Example 77: In the chart to the right, selected values of x are given along with values of $f(x)$ and $f'(x)$.

x	$f(x)$	$f'(x)$
-2	3	-1
-1	2	-2
0	-1	4
1	1	-3

a) If f^{-1} is the inverse function of f, find the derivative of f^{-1} at $x=-1$: $\left(f^{-1}\right)'(-1)$.

In this problem, it is probably better to know the formula for the derivative of the inverse.

$f(-1)=2$ (point is $(-1,2)$) so $f^{-1}(2)=-1$ (point on inverse is $(2,-1)$)

$$\text{By formula}: \left(f^{-1}\right)'(-1) = \frac{1}{f'(y)} = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{1}{f'(-1)} = \frac{1}{-2} = \frac{-1}{2}$$

b) Write an equation for the line tangent to the graph of $y = f^{-1}(x)$ at $x = 2$.

From part a), $m = \frac{-1}{2}$, point is $(2,-1)$: Tan line: $y+1 = \frac{-1}{2}(x-2)$ or $y = \frac{-x}{2}$

V. Differential Equations

What you are finding: A differential equation (DEQ) is in the form of $\frac{dy}{dx} = (\text{Algebraic Expression})$. The goal of solving a DEQ is to work backwards from the derivative to the function; that is to write an equation in the form of $y = f(x) + C$ (since the technique involves integration, the general solution will have a constant of integration). If the value of the function at some value of x is known (an initial condition), the value of C can be found. This is called a specific solution.

How to find it: In Calculus AB, the only types of DEQ's studied are called separable. Separable DEQ's are those than that can be in the form of $f(y)\,dy = g(x)\,dx$. Once in that form, both sides can be integrated with the constant of integration on only one side, usually the right.

Word problems involving change with respect to time are usually models of DEQ's. A favorite type is a problem using the words: The rate of change of y is proportional to some expression. The equation that describes this statement is: $\frac{dy}{dt} = k \cdot (\text{expression})$. The rate of change is usually with respect to time.
Usually there is a problem that requires you to create a slope field. Simply calculate the slopes using the given derivative formula and plot them on the given graph. Usually the slopes will be integer values.

Example 78: Consider the differential equation $\frac{dy}{dx} = (y+2)^2 \sin\left(\frac{x}{e}\right)$

a) There is a horizontal line with equation $y = k$ that satisfies the differential equation. Find the value of k.

Since the line is horizontal, $\frac{dy}{dx} = 0$, so $k = -2$.

b) There is a vertical line in the form of $x = w$, such that for any value of w, the function $y = f(x)$ has a horizontal tangent line. Find the smallest positive value of w.

$\sin\left(\frac{w}{e}\right) = 0 \Rightarrow \frac{w}{e} = \pi, 2\pi, 3\pi \ldots \Rightarrow$ Smallest value of $w = \pi e$.

c) Find the particular solution $y = f(x)$ with the initial condition $f(0) = -1$.

$$\int \frac{dy}{(y+2)^2} = \int \sin\left(\frac{x}{e}\right) dx$$

$$\frac{-1}{y+2} = -e\cos\left(\frac{x}{e}\right) + C$$

$$\frac{-1}{-1+2} = -e\cos(0) + C \Rightarrow -1 = -e + C \Rightarrow C = e - 1$$

$$\frac{-1}{y+2} = e\cos\left(\frac{x}{e}\right) + e - 1$$

$$y + 2 = \frac{-1}{e\cos\left(\frac{x}{e}\right) + e - 1} \Rightarrow y = -2 - \frac{1}{e\cos\left(\frac{x}{e}\right) + e - 1}$$

Example 79: Consider the differential equation $\dfrac{dy}{dx}=\dfrac{-4x+2}{y}$.

a) Let $y=f(x)$ be the particular solution to the differential equation with the initial condition $f(2)=-4$. Write an equation for the line tangent to the graph at (2, -4) and use it to approximate $f(1.8)$.

$$\frac{dy}{dx}=\frac{-8+2}{-4}=\frac{-6}{-4}=\frac{3}{2}$$

$$\text{Tangent line eq}: y+4=\frac{3}{2}(x-2) \text{ or } y=\frac{3x}{2}-7 \text{ so } f(1.8)\approx -4.3$$

b) Find the particular solution $y=f(x)$ to the differential equation with the initial condition $f(2)=-4$.

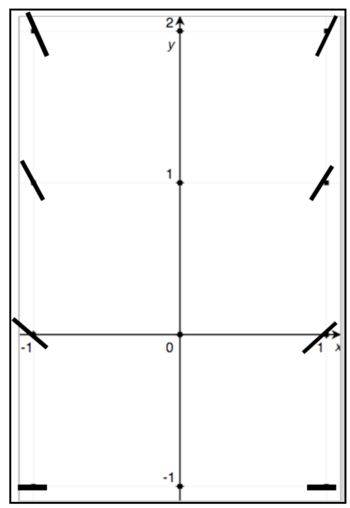
$$\int y\,dy=\int(-4x+2)\,dx$$

$$\frac{y^2}{2}=-2x^2+2x+C \qquad \frac{16}{2}=-2(4)+2(2)+C\Rightarrow C=12$$

$$\frac{y^2}{2}=-2x^2+2x+12\Rightarrow y^2=-4x^2+4x+24\Rightarrow y=\pm2\sqrt{-x^2+x+6}$$

Since the solution goes through $(2,-4)$, the solution is $y=-2\sqrt{-x^2+x+6}$

Example 80: Consider the differential equation $\dfrac{dy}{dx}=\dfrac{y+1}{x^3}, x\neq 0$.



a) On the axes provided, sketch a slope field for this DEQ.

b) Find the particular solution $y=f(x)$ to the differential equation given that $f(1)=0$.

$$x^3dy=(y+1)\,dx\Rightarrow\frac{dy}{y+1}=\frac{dx}{x^3}$$

$$\int\frac{dy}{y+1}=\int\frac{dx}{x^3}\Rightarrow \ln|y+1|=\frac{-1}{2x^2}+C$$

$$y+1=Ce^{\frac{-1}{x^2}}\Rightarrow y=-1+Ce^{\frac{-1}{2x^2}}$$

$$x=1, y=0\Rightarrow 0=-1+Ce^{-1/2}\Rightarrow\frac{C}{\sqrt{e}}=1\Rightarrow C=\sqrt{e}$$

$$y=-1+\sqrt{e}\cdot e^{\frac{-1}{x^2}} \text{ or } y=-1+e^{\frac{1}{2}-\frac{1}{x^2}} \text{ or } y=-1+e^{\frac{x^2-2}{2x^2}}$$

c) For the solution found in b), find $\lim\limits_{x\to-\infty}f(x)$.

$$\lim_{x\to-\infty}\left(-1+e^{1-\frac{1}{x^2}}\right)=\sqrt{e}-1$$

Example 81: (Calc) Let $R(t)$ represent the number of people in an office building who have heard a rumor at time *t* minutes where $t \geq 0$. $R(t)$ is increasing at a rate proportional to $160 - 2R(t)$, where the constant of proportionality is *k*.

a) If 10 people start the rumor, find $R(t)$ in terms of *t* and *k*.

$$\frac{dR}{dt} = k(160 - 2R)$$
$$\frac{dR}{160 - 2R} = k\,dt \Rightarrow \int \frac{dR}{2R - 160} = \int -k\,dt$$
$$\frac{1}{2}\ln|2R - 160| = -kt + C \Rightarrow \ln|2R - 160| = -2kt + C$$
$$2R - 160 = Ce^{-2kt} \Rightarrow R = \frac{160 + Ce^{-2kt}}{2}$$
$$t = 0, R = 10 \Rightarrow 10 = \frac{160 + C}{2} \Rightarrow C = -140$$
$$R = \frac{160 - 140e^{-2kt}}{2} = 80 - 70e^{-2kt}$$

b) If 40 people know the rumor 5 minutes after it started, find $R(t)$ in terms of *t*.

$$t = 5, R = 40 \Rightarrow 40 = 80 - 70e^{-10k}$$
$$40 = 80 - 70e^{-10k} \Rightarrow 70e^{-10k} = 40 \Rightarrow e^{-10k} = \frac{40}{70}$$
$$-10k = \ln\left(\frac{40}{70}\right) \Rightarrow k = \ln\left(\frac{4}{7}\right)\Big/-10 = 0.056$$
$$R = 80 - 70e^{-0.112t}$$

c) To the nearest person, how many people in the office have heard the rumor 10 minutes after it started?

$$R = 80 - 70e^{-.112(10)} \approx 57 \text{ people}$$

d) Find $\lim\limits_{t \to \infty} R(t)$.

$$\lim_{t \to \infty} R(t) = \lim_{t \to \infty} \left(80 - \frac{70}{e^{2kt}}\right) = 80.$$

e) How many people have heard the rumor when the rumor is growing the fastest? Justify your answer.

$$\frac{dR}{dt} = k(160 - 2R) = 0 \Rightarrow 160 - 2R = 0 \Rightarrow R = 80 \text{ people}$$

Example 82: (Calc) At time t =0 a biker is traveling at 900 ft/min when he decides not to do any pedaling. He slows down with a negative acceleration that is directly proportional to t^2. This brings the biker to a stop in 3 minutes.

a) Write an expression for the velocity of the biker at time t.

$$a(t) = -kt^2$$
$$v(t) = \int -kt^2 dt = \frac{-kt^3}{3} + C$$
$$v(0) = 0 + C = 900 \Rightarrow v(t) = 900 - \frac{kt^3}{3}$$
$$\text{Stop}: v(3) = 0 = 900 - \frac{k(3)^3}{3} \Rightarrow 9k = 900 \Rightarrow k = 100$$
$$v(t) = 900 - \frac{100t^3}{3}$$

b) How far does the biker coast in that 3 minute time period?

$$\text{Since velocity } > 0\text{, Distance} = \int_0^3 \left(900 - \frac{100t^3}{3}\right) dt$$
$$= \left(900t - \frac{100t^4}{12}\right)\Big|_0^3 = 2025 \text{ ft}$$

Example 83: Consider the differential equation $xy' - y = 4x^2y, \quad x > 0$.

a) Find the general solution of the DEQ.

$$x\frac{dy}{dx} = y + 4x^2y \quad \Rightarrow \quad x\frac{dy}{dx} = y(4x^2 + 1)$$
$$\frac{dy}{y} = \frac{4x^2 + 1}{x}dx$$
$$\int \frac{dy}{y} = \int \left(4x + \frac{1}{x}\right) dx$$
$$\ln|y| = 2x^2 + \ln|x| + C$$
$$y = Ce^{2x^2 + \ln|x|} = Ce^{2x^2} \cdot e^{\ln x} = Cxe^{2x^2}$$
$$y = Cxe^{2x^2}$$

b) Find the particular solution of the DEQ passing through (1,1).

$$1 = Ce^2 \Rightarrow C = \frac{1}{e^2} = e^{-2} \Rightarrow y = e^{-2}xe^{2x^2} \Rightarrow y = xe^{2x^2-2}$$